Ex Compute $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

Sol. $\sum_{n=-\infty}^{\infty} \frac{1}{n^2+1} = -\left(\operatorname{Res}\left(\frac{\pi \cot \pi z}{z^2+1}, i\right) + \operatorname{Res}\left(\frac{\pi \cot \pi z}{z^2+1}, -i\right)\right)$

Let $g(z) = \pi \cot \pi z$ and $h(z) = z^2+1$. Then

$$\operatorname{Res}\left(\frac{g}{h}, i\right) = \frac{g(i)}{h'(i)} = \frac{\pi \cot(\pi i)}{2i} = \frac{\pi}{2i}\left(\frac{\frac{e^{ii\pi}+e^{-ii\pi}}{2}}{\frac{e^{ii\pi}-e^{-ii\pi}}{2i}}\right) = \frac{\pi}{2}\left(\frac{e^{-\pi}+e^{\pi}}{e^{-\pi}-e^{\pi}}\right)$$

$\operatorname{Res}\left(\frac{g}{h}, -i\right)$ = the same

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2+1} = -\pi \frac{e^{-\pi}+e^{\pi}}{e^{-\pi}-e^{\pi}} = \pi \frac{e^{+\pi}+e^{-\pi}}{e^{\pi}-e^{-\pi}} = \pi \frac{e^{2\pi}+1}{e^{2\pi}-1}$$

Thus, $\sum_{n=1}^{\infty} \frac{1}{n^2+1} = \frac{1}{2}\left(\pi \frac{e^{2\pi}+1}{e^{2\pi}-1} - 1\right) \approx 1.076674$

Note: This result can be generalized as follows.

$$\sum_{n=1}^{\infty} \frac{1}{n^2+a^2} = \frac{a\pi \coth(a\pi) - 1}{2a^2}, \quad a > 0.$$

Just follow the steps above, keeping track of the a, and recall that

$$\coth(x) = \frac{e^x+e^{-x}}{e^x-e^{-x}} = \frac{\cosh x}{\sinh x}$$

Ex Show that $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$. This was first "shown" by Euler in 1735.

Sol. You book does this using a different method. Here we will show that $\lim_{a\to0^+}\left(\sum_{n=1}^{\infty}\frac{1}{n^2+a^2}\right)=\frac{\pi^2}{6}$.

$$\lim_{a\to0^+}\sum_{n=1}^{\infty}\frac{1}{n^2+a^2}=\lim_{a\to0^+}\frac{\pi a\coth\pi a-1}{2a^2}=\frac{1}{2}\lim_{a\to0^+}\frac{\pi a\cosh\pi a-\sinh\pi a}{a^2\sinh\pi a}$$

This has indeterminent form $\frac{0}{0}$, so we can apply L'Hospital's Rule.

$$=\frac{1}{2}\lim_{a\to0^+}\frac{\pi\,\cancel{\cosh\pi a}+\pi^2\cancel{a}\sinh\pi a-\pi\,\cancel{\cosh\pi a}}{2\cancel{a}\sinh\pi a+\pi a^{\cancel{2}}\cosh\pi a}=\frac{\pi^2}{2}\lim_{a\to0^+}\frac{\sinh\pi a}{2\sinh\pi a+\pi a\cosh\pi a}$$

$\to\frac{0}{0}$

$$=\frac{\pi^2}{2}\lim_{a\to0^+}\frac{\pi\cosh\pi a}{2\pi\cosh\pi a+\pi\cosh\pi a+\pi^2 a\sinh\pi a}=\frac{\pi^2}{2}\cdot\frac{\pi}{3\pi+0}$$

$$=\frac{\pi^2}{6}.$$

Look up the "Basel problem" in Wikipedia to see how Euler did it.