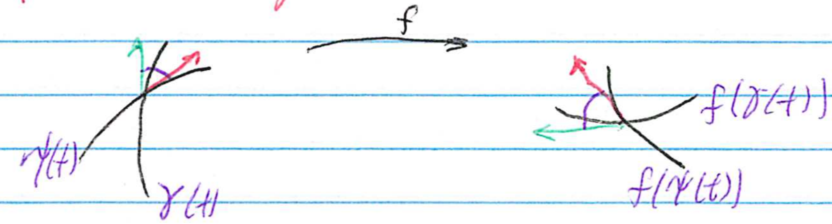
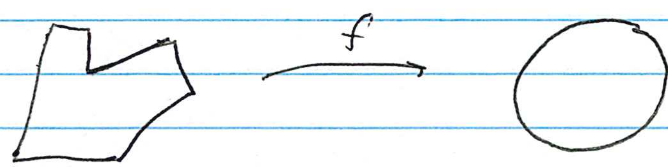
Ch 5 Conformal Mappings

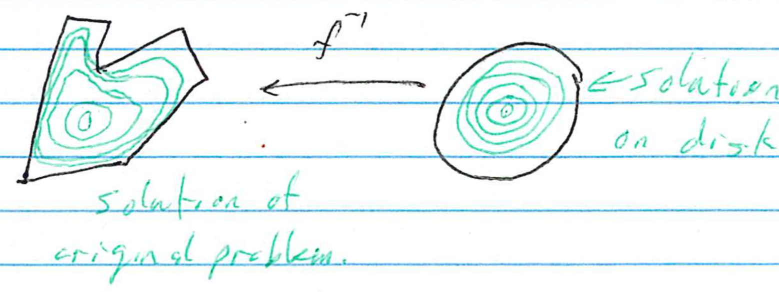
5.1 Recall: Let $A, B \subset \mathbb{C}$. A map $f: A \to B$ is <u>conformal</u> if it preserves angles:



See pages 64-65 in Section 1.5. It was shown (Thm 1.5.7 or Thm 5.1.1) that if f is analytic and $f'(z) \neq 0$ $\forall z \in A$, then f is conformal.

Conformal maps are used in many physical problems. The idea is if you need to solve some problem on a complicated set, you find a conformal map to a simple set, like a disk, solve the problem there, and then pull back the solution to the set you started with.





The (upcoming) Riemann Mapping Theorem will tell us that f exists. Actually finding f can be hard.

5.1 is about existence of f.

5.2 is tricks for finding f (sometimes)

5.3 is applications.

Prop 5.1.2 has two quick facts ...

① Let $f: A \to B$ be conformal and bijective. Then f^{-1} exists and is also conformal.

② Let $f: A \to B$ and $g: B \to C$ be conformal and bijective. Then so is $g \circ f: A \to C$.

The proof of ① comes from the inverse function thm (1.5.10)

$$\frac{d f^{-1}}{dz} = \frac{1}{f'(f^{-1}(z))} \quad \begin{matrix} \leftarrow \text{never zero!} \\ \leftarrow \text{never zero} \end{matrix}$$

and ② comes from the chain rule.

Note: Let $A \subset \mathbb{C}$ and $\mathcal{C}(A)$ = all conformal bijective maps from A to A. The $\mathcal{C}(A)$ is a group under composition.

Prop 5.1.3 Let u be harmonic on a region $B \subset \mathbb{R}^2 = \mathbb{C}$, and suppose $f: A \to B$ is analytic. Then $u \circ f$ is harmonic on A.

Pf: Let $z \in A$, $w = f(z) \in B$ and let U be open disk, $w \in U \subset B$. Let $V = f^{-1}(U)$ [f need not be one-to-one] We will show $u \circ f$ is harmonic on V. This suffices since $z \in A$ was arbitrary. $\exists$ analytic function $g: U \to \mathbb{C}$ s.t. $u = \operatorname{Re} g$. Then

$$\operatorname{Re}(g \circ f) = \operatorname{Re}\left(u(f) + i v(f)\right) = u \circ f \qquad (v = \operatorname{Im} g).$$

Since $g \circ f$ is analytic (chain rule) $\operatorname{Re}(g \circ f)$ is harmonic.

Example. Let $u(x,y) = x^2 - y^2$ and $f(z) = z^3$. Check that $w = u \circ f$ is harmonic, $\nabla^2 w = 0$.

$f(x+iy) = (x+iy)^3 = x^3 + 3x^2yi - 3xy^2 - y^3i = (x^3 - 3xy^2) + (3x^2y - y^3)i.$

Thus $u(f(x+iy)) = [x^3 - 3xy^2]^2 - [3x^2y - y^3]^2$

$$= x^6 - 6x^4y^2 + 9x^2y^4 - (9x^4y^2 - 6x^2y^4 + y^6)$$

$$= x^6 - 15x^4y^2 + 15x^2y^4 - y^6 = w.$$

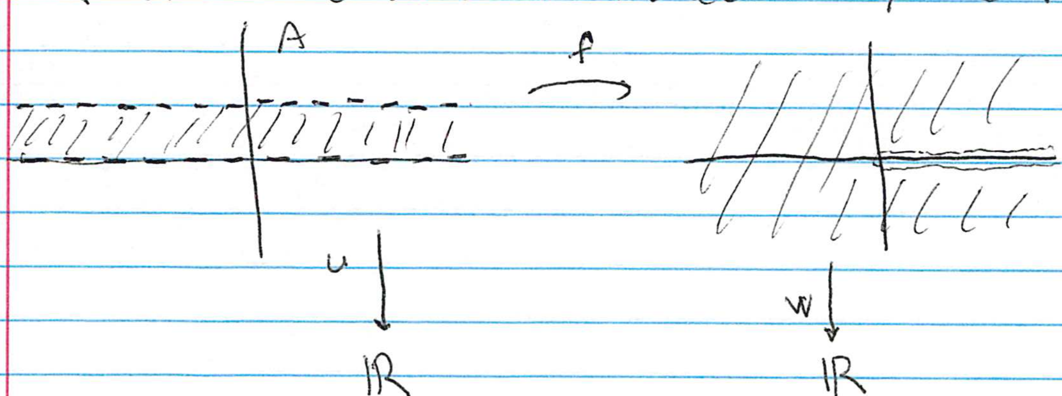
$w_x = 6x^5 - 60x^3y^2 + 30xy^4 - 0$ $\quad w_{xx} = 30x^4 - 180x^2y^2 + 30y^4$

$w_y = 0 - 30x^4y + 60x^2y^3 - 6y^5$ $\quad w_{yy} = -30x^4 + 180x^2y^2 - 30y^4$

Thus, $w_{xx} + w_{xy} = 0$ so, $w = u \circ f$ is harmonic.

Example (5.1.8 in textbook) Let $A = \{z \in \mathbb{C} \mid 0 < \operatorname{Im} z < 2\pi\}$ and $B = \mathbb{C} - \{\text{nonneg. real axis}\} = \mathbb{C} - \{x + i0 \mid x \geq 0\}$.
Let $f(z) = e^z$. Then $f: A \to B$ is a bijective conformal map. [For bijective, review section 1.3 if you need to. Recall $(e^z)' = e^z$ is never 0.

Let $u(x,y) = x + y$ for $x + iy \in A$. Clearly, $\nabla^2 u = 0$ on A, so u is harmonic on A. Find the corresponding harmonic function on B induced by the transform f.



What this means is find $w: B \to \mathbb{R}$ s.t. $\nabla^2 w = 0$ and

$$w = u \circ f^{-1}.$$

We know f^{-1} is analytic. Thus $u \circ f^{-1}$ is harmonic on B.

Now $\quad f^{-1}(x+iy) = \log(x+iy) = \ln|x+iy| + i\arg(x+iy)$

$$= \ln\sqrt{x^2+y^2} + i\arctan\left(\frac{y}{x}\right)$$

Then $\quad u \circ f^{-1}(x+iy) = u\left(\operatorname{Re} f^{-1}(x+iy), \operatorname{Im}(f^{-1}(x+iy))\right) =$

$$\ln\sqrt{x^2+y^2} + \arctan(y/x)$$

Thus, $\quad w(x,y) = \frac{1}{2}\ln(x^2+y^2) + \arctan(y/x)$.

By Prop 5.1.3. $w(x,y)$ is harmonic, $\nabla^2 w = 0$, but you may wish to check this directly.

Now we came to the Riemann mapping theorem.

The proof is beyong the scope of thos course, but is in the Internet Supplement.

The RMT is abrat open sets. ~~Another~~ Two other thms consider the boundary.

RMT Let A be an open, connected, simply connected, proper subset of $\mathbb{C}$. (proper means $A \neq \emptyset$, $A \neq \mathbb{C}$) (simply connected means "no holes.") (A can be bounded or unbounded.) Then

$$\exists\ f: A \to D(0,1) = \text{open unit disk}, |z|<1 \subset \mathbb{C}$$

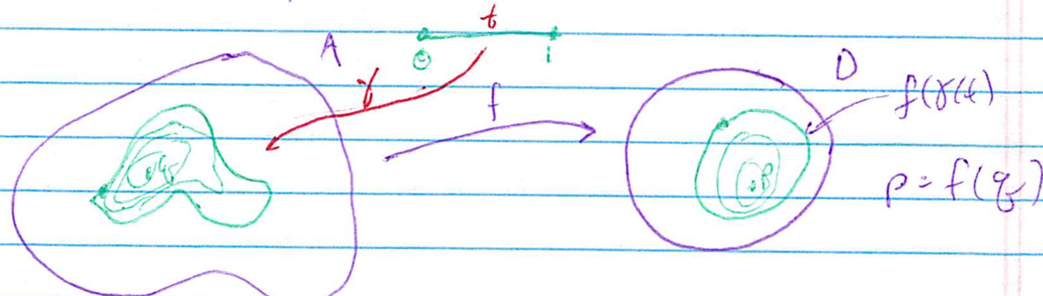
that is bijective and conformal.

If we specify that for some $z_0 \in A$, that $f(z_0)=0$ and $f'(z_0)>0$, then f is unique. (see also Exercise 7.)

Note: Pf of uniqueness is in textbook. Read if you wish. Full proof is in the Internet Supplement, and other references given in the textbook.

The condition that $A \neq \mathbb{C}$ is necessary. If $f: \mathbb{C} \to D$ the clearly $|f|$ is bounded. But f is entire (analytic on $\mathbb{C}$, so Liouville's Thm[2.4.8] say f is a constant function. So, f cannot be one-to-one or onto! (See Exercise 11)

The condition that A be simply connected is necessary. In the unit disk every closed loop is homotopic to a point. If we let $\gamma(t)$ be a loop in A and $f: A \to$ unit disk is a conf. bij. with cont. inverse, then the homotopy map that takes $f(\gamma(t))$ to a pt in D can be pulled back to give a homotopy of $\gamma(t)$ to a pt in A.



If $H: I \times I \to D$ has $H(0,t) = f(\gamma(t))$, $H(1,t) = p$, then $f^{-1}(H(0,t)) = \gamma(t)$ and $f^{-1}(H(1,t)) = f^{-1}(p) = q$.

There are many examples of conformal bijections between bounded and unbounded regions. We will study several in 5.2 See pages 340-341.

Next we consider the boundaries of the regions in question.

① Let A and B be two open conn'd subsets of $\mathbb{C}$ (regions). Suppose $f: A \to f(A)$ is conformal, and boundary of $f(A)$ is the same as the boundary of B.

$$\text{bd } f(A) = \text{bd } B.$$

Then $f(A)$ is either $= B$ or two $\mathbb{C} - \text{cl } B$. This follows from the fact that $f(A)$ must be conn'd. See pages 322-323 for details. Thus, you can use one test point to see which it is: let $z_0 \in A$. If $f(z_0) \in B$, then $f(A) = B$. If $f(z_0) \in \mathbb{C} - \text{cl}(B)$, then $f(A) = \mathbb{C} - \text{cl}(B)$.

② Thm 5.1.5 (Osgood-Caratheodory Thm) Let A_1 and A_2 be bdd simply conn'd regions (open conn'd) whose boundaries are simple cont. closed curves γ_1, γ_2 resp.

Let $f: A_1 \to A_2$ be a conformal bijection.

Then f can be extended to a continuous map $\tilde{f}: A_1 \cup \gamma_1 \to A_2 \cup \gamma_2$ this is also a bijection.

③ Thm 5.1.6 Let A be a bounded region (open conn'd) with

$$f: A \to f(A) \subset \mathbb{C}$$

a bij. conf. map. Assume that f can be extended to a cont. map on $cl(A)$ and that $f(bd(A))$ is a circle of radius R. Then $f(A)$ is the disk inside that circle.

Pf: See textbook. Just use ① & ②, and the Max. Modulus Thm.