5.2

Def Let $T(z) = \frac{az+b}{cz+d}$, $ad-bc \neq 0$. Then $T(z)$ is called a Möbius function or a Fractional linear transformation.

Properties (5.2.1) Domain $= \mathbb{C} - \{-d/c\}$. Range $= \mathbb{C} - \{\frac{a}{c}\}$.

T is a bijection. Its inverse is $T^{-1}(w) = \frac{-dw+b}{cw-a}$.

T is analytic and conformal on its domain.

$$T' = \frac{ad-bc}{(cz+d)^2}.$$

Recall $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the Riemann Sphere. A line in $\mathbb{C}$ can be thought of as a circle in $\overline{\mathbb{C}}$ going through ∞.

Prop. (5.2.3) Möbius function take circles to circle in $\overline{\mathbb{C}}$, or in $\mathbb{C}$ they take lines to lines or circles and circles to circles or lines.

Pf We can decompose $T(z) = \frac{az+b}{cz+d}$ into, $T = T_4 \circ T_3 \circ T_2 \circ T_1$ where

$$T_1(z) = z + \frac{d}{c} \quad T_2(z) = \frac{1}{z}, \quad T_3(z) = (bc-ad)\,z/c^2$$

and $T_4(z) = z + a/c$, assuming $c \neq 0$. (Exercise 11)

If $c=0$, $T(z) = \frac{a}{d}z + \frac{b}{d}$ is linear, and thus takes lines to lines and circles to circles. Clearly, T_1, T_3 and T_4 take lines to lines and circles to circles.

We need to check $T_2(z) = \frac{1}{z}$.

An equation of this form

$$\star \quad Ax + By + C(x^2+y^2) = D, \quad A, B, C \text{ not all } 0,$$

determines either a line $(C=0)$ or a circle $(C \neq 0)$ in the xy-plane. In fact, any line or circle can be expressed in this form.

Pf ~~If C=0~~ Any line can be expressed as $Ax + By = D$.

Any circle can be expressed as

$$(x-h)^2 + (y-k)^2 = R^2$$

or $$-2hx - 2ky + x^2+y^2 = R^2 - h^2 - k^2.$$

The equation $\star$, $C \neq 0$, can be rewritten as

$$\left(x - \frac{A}{2C}\right)^2 + \left(y - \frac{B}{2C}\right)^2 = \frac{D}{C} - \frac{A^2}{4C^2} - \frac{B^2}{4C^2}$$

which is a circle.

Let $z = x+iy \neq 0$. Then $\frac{1}{z} = \frac{1}{x+iy}\frac{x-iy}{x-iy} = \frac{x}{x^2+y^2} - \frac{iy}{x^2+y^2}$

$$= u + iv.$$

Since $z = \frac{1}{\frac{1}{z}}$ we also have $x = \frac{u}{u^2+v^2}$, $y = \frac{-v}{u^2+v^2}$

and $x^2+y^2 = \frac{1}{u^2+v^2}$. Thus

$\star$ becomes $$\frac{Au}{u^2+v^2} - \frac{Bv}{u^2+v^2} + \frac{C}{u^2+v^2} = D$$

$$\Rightarrow \quad Au - Bv - D(u^2+v^2) = -C,$$ which is a circle or a line. $\square$

Prop 5.2.2 Let D = open unit disk ($|z|<1$). Let $T(z) = e^{i\theta}\dfrac{z-z_0}{1-\bar{z}_0 z}$ for some $z_0 \in D$ and $\theta = [0, 2\pi)$. Then T is a conformal map of D onto itself. Furthermore, any conformal map of D onto itself is of this form.

Note For $\theta = \pi$, $T(0) = z_0$ and $T(z_0) = 0$.

Pf $T(z)$ is analytic since the only sing. is outside $cl(D)$. $\left(\left|\frac{1}{\bar{z}}\right| = \frac{1}{|z|} > 1\right)$

We can extend T to $bd(D)$ and show $T: bd\, D \to bd\, D$.

Let $|z|=1$. $$|T(z)| = \left|\frac{z-z_0}{1-\bar{z}_0 z}\right| = \frac{|z-z_0|}{\underbrace{|z|}_{1}\,|z^{-1}-\bar{z}_0|} = \frac{|z-z_0|}{|\bar{z}-\bar{z}_0|} =$$

$$\frac{|z-z_0|}{|\overline{z-z_0}|} = 1.$$

By the Maximum Modulus Thm (2.5.6) the max of $|T(z)|$ occurs on the $bd(D)$. Thus T maps D into D.

Recall in general: $T^{-1}(w) = \dfrac{-dw+b}{cw-a}$ for $w = T(z) = \dfrac{az+b}{cz+d}$.

Apply this to $T(z) = \dfrac{e^{i\theta}z - e^{i\theta}z_0}{1-\bar{z}_0 z}$ to get

$T^{-1}(w) =$

$$T^{-1}(w) = \frac{-w - e^{i\theta}z_0}{-\bar{z}_0 w - e^{i\theta}} = \frac{-(w + e^{i\theta}z_0)}{-e^{i\theta}(1 + e^{-i\theta}\bar{z}_0 w)}$$ Let $w_0 = -e^{i\theta}z_0$.

$$= e^{-i\theta}\,\frac{w - w_0}{1 - \bar{w}_0 w}.$$

This is the same form as T. Thus T^{-1} maps D onto D. Thus T was onto.

$\blacksquare$

You may recall that three points in the plane determine a circle (or a line). Extra Credit: Prove this! The next proposition we use this to gain control over where a Möbius functions send a give circl or line.

Prop 5.2.4 (Cross Ratios) Let $\{z_1, z_2, z_3\}$ and $\{w_1, w_2, w_3\}$ be two sets of distinct points in $\mathbb{C}$. Then there is a unique Möbius function T such that $T(z_i) = w_i$, $i = 1, 2, 3$. The formula for T can be derived from

$$\left(\frac{w - w_1}{w - w_2}\right)\left(\frac{w_3 - w_2}{w_3 - w_1}\right) = \left(\frac{z - z_1}{z - z_2}\right)\left(\frac{z_3 - z_2}{z_3 - z_1}\right)$$

Outline of Pf It is easy to check this eq holds for $w = w_1$, $z = z_1$, $w = w_2$, $z = z_2$ (allowing ∞) and $w = w_3$, $z = z_3$.

To get $w = T(z)$ just solve for w.

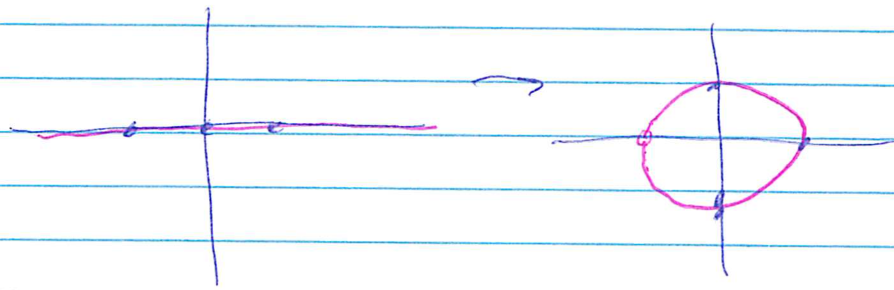
$$\frac{w - w_1}{w - w_2} = \left(\frac{z - z_1}{z - z_2}\right)\left(\frac{z_3 - z_2}{z_3 - z_1}\right)\left(\frac{w_3 - w_1}{w_3 - w_2}\right) = Q(z).$$

Then $$w = \frac{-w_2 Q + w_1}{-Q + 1} = T(z).$$

Ex Find $T(z)$ s.t. $T(-1)=i$, $T(0)=1$, $T(1)=-i$.

That is $z_1=-1$, $z_2=0$, $z_3=1$

$w_1=i \quad w_2=1 \quad w_2=-i$.



$$\frac{w-i}{w-1}\left(\frac{-i-1}{-i-i}\right)=\frac{z+1}{z}\left(\frac{1-0}{1-(-1)}\right)$$

$$\frac{w-i}{w-1}\,\frac{1+i}{2i}=\frac{z+1}{z}\,\frac{1}{2}$$

$$\frac{w-i}{w-1}=\frac{z+1}{z}\,\frac{1}{2}\cdot\frac{2i}{1+i}=\frac{z+1}{z}\left(\frac{i}{1+i}\right)\cdot\frac{(1-i)}{(1-i)}=\frac{z+1}{z}\underbrace{\left(\frac{1+i}{2}\right)}_{=\alpha}$$

$$w-i=\frac{z+1}{z}\cdot\alpha\cdot(w-1)=\alpha\left(\frac{z+1}{z}\right)w-\alpha\left(\frac{z+1}{z}\right)$$

$$w-\alpha\left(\frac{z+1}{z}\right)w=i-\alpha\left(\frac{z+1}{z}\right)$$

$$w=\frac{i-\alpha\left(\frac{z+1}{z}\right)}{1-\alpha\left(\frac{z+1}{z}\right)}\cdot\frac{z}{z}=\frac{(i-\alpha)z-\alpha}{(1-\alpha)z-\alpha}$$

$$=\frac{\left(\frac{-1+i}{2}\right)z-\frac{1+i}{2}}{\left(\frac{1-i}{2}\right)z-\frac{1+i}{2}}\cdot\frac{2}{2}=\frac{(-1+i)z-(1+i)}{(1-i)z-(1+i)}.$$

Also. $T(\infty)=-1$, $T(i)=\infty$, $T(-i)=0$. $\square$

Möbius functions have many other interesting properties.

You can read in the textbook about reflections through a circle. See Exercise 35 for another direction.

Next we do some interesting examples of other types of conformal maps.