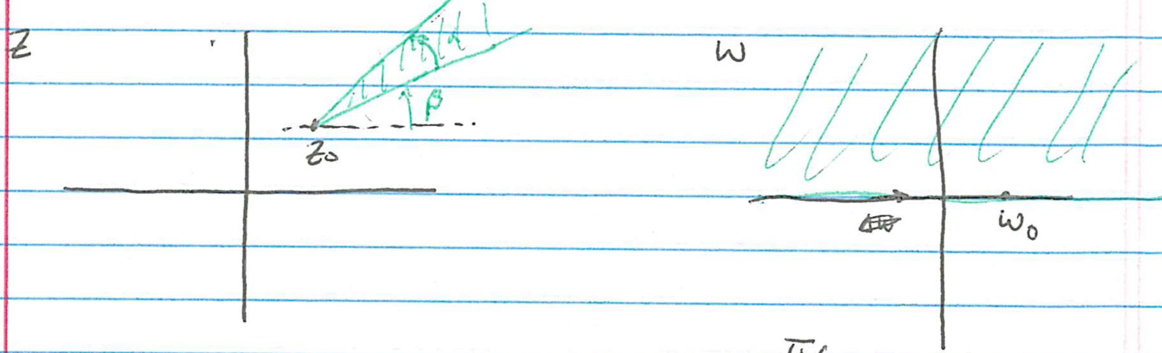
Example Find a map $w=f(z)$ that take the sector below to the upper half plan with $z_0 \to w_0$.



Answer: $w = w_0 + \left[e^{-i\beta}(z-z_0)\right]^{\pi/\alpha}$

Think about the inverse: $z = f^{-1}(w) = e^{i\beta}(w-w_0)^{\alpha/\pi} + z_0$

Then $$\frac{dz}{dw} = \frac{\alpha}{\pi} e^{i\beta}(w-w_0)^{\frac{\alpha}{\pi}-1}$$

~~arg Δw → 0~~ ~~Move along real axis by Δw → 0.~~

~~arg Δw → 0~~

$\arg \frac{dz}{dw}$ as we move along real axis is

$$= \beta + \left(\frac{\alpha}{\pi}-1\right)\arg(w-w_0)$$

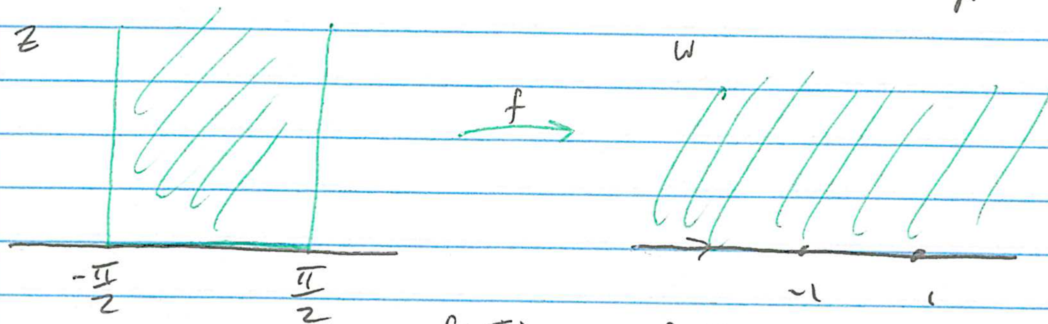
$$\boxed{\begin{aligned}&\arg re^{i\theta} = \theta\\ &\arg re^{i\theta}e^{i\alpha}\\ &= \theta+\alpha\end{aligned}}$$

$\arg (re^{i\theta})^x = x\theta.$

$$\arg(w-w_0) = \begin{cases} \pi & w < w_0 \\ 0 & w > w_0 \end{cases}$$

$$\arg \frac{dz}{dw} = \begin{cases} \beta+\alpha-\pi & w < w_0 \\ \beta & w > w_0 \end{cases}$$

Example   Find a map that does the following



$$f(-\tfrac{\pi}{2}) = -1, \quad f(\tfrac{\pi}{2}) = 1$$

We will find f^{-1}. What can we say about $\arg \frac{dz}{dw}$?

$$\arg \frac{dz}{dw} = \begin{cases} -\frac{\pi}{2} & w < -1 \\ 0 & -1 < w < 1 \\ \frac{\pi}{2} & w > 1 \end{cases}$$

"Guess" $\quad \dfrac{dz}{dw} = A\,(w+1)^{-\frac{1}{2}}(w-1)^{-\frac{1}{2}}.$

Now $\quad \arg \dfrac{dz}{dw} = \arg A - \frac{1}{2}\arg(w+1) - \frac{1}{2}(w-1)$

$$\arg(w+1) = \begin{cases} \pi & w < -1 \\ 0 & w > -1 \end{cases}$$

$$\arg(w-1) = \begin{cases} \pi & w < 1 \\ 0 & w > 1 \end{cases}$$

Now $\quad \arg\left(\dfrac{dz}{dw}\right) = \arg A - \dfrac{1}{2}\begin{cases} 2\pi & w < -1 \\ \pi & -1 < w < 1 \\ 0 & w > 1 \end{cases}$

If $\arg A = \frac{\pi}{2}$ we have it.

$$\frac{dz}{dw} = e^{i\frac{\pi}{2}}(w+1)^{-\frac{1}{2}}(w-1)^{-\frac{1}{2}} = \frac{i}{\sqrt{w^2-1}}$$

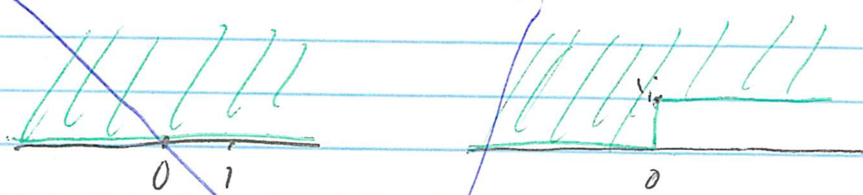
$$= \frac{1}{\sqrt{1-w^2}}$$

Thus $z = \int_0^w \frac{1}{\sqrt{1-w^2}}\, dw = \arcsin(w)$.

Thus $f(z) = \sin z$.

Skip: done separately.

<u>Example</u> Map upper half plane to



$$\arg \frac{dz}{dw} = \begin{cases} 0 & w < 0 \\ -\pi/2 & 0 \le w < 1 \\ 0 & w > 1 \end{cases}$$

$$\frac{dz}{dw} = A\, w^{-\frac{1}{2}} (w-1)^{\frac{1}{2}}$$

$$\arg \frac{dz}{dw} = \arg A - \frac{1}{2} \underbrace{\arg w}_{\begin{cases} \pi & w<0 \\ 0 & w>0 \end{cases}} + \frac{1}{2} \underbrace{\arg (w-1)}_{\begin{cases} \pi & w<1 \\ 0 & w>1 \end{cases}}$$

$$\Rightarrow \arg A = 0$$

$$\frac{dz}{dw} = \sqrt{\frac{w-1}{w}}$$

$$z = \int_0^w \sqrt{\frac{w-1}{w}}\, dw + i$$

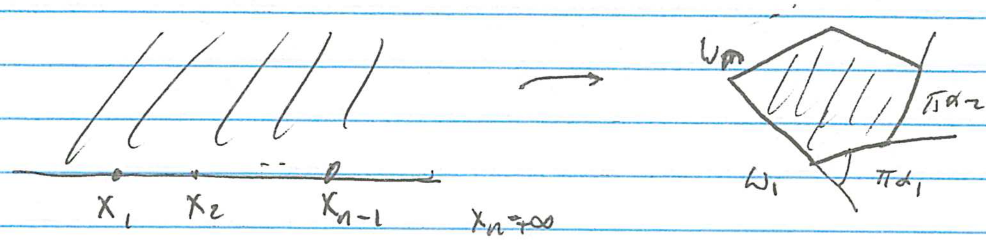
(S64 p 220)

$\alpha_i \in (-1,1)$

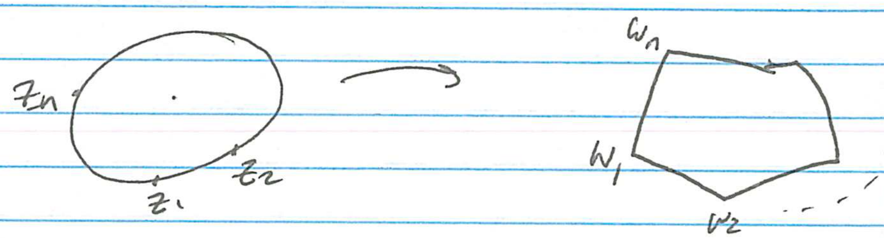
Schwarz-Christoffel Thm (5.2.11) $\quad \sum_{i=1}^{n} \alpha_i \pi = 2\pi$

$$f(z) = a \int_{z_0 = 0}^{z} (w - x_1)^{-\alpha_1} (w - x_2)^{-\alpha_2} \cdots (w - x_{n-1})^{-\alpha_{n-1}} \, dw + b$$

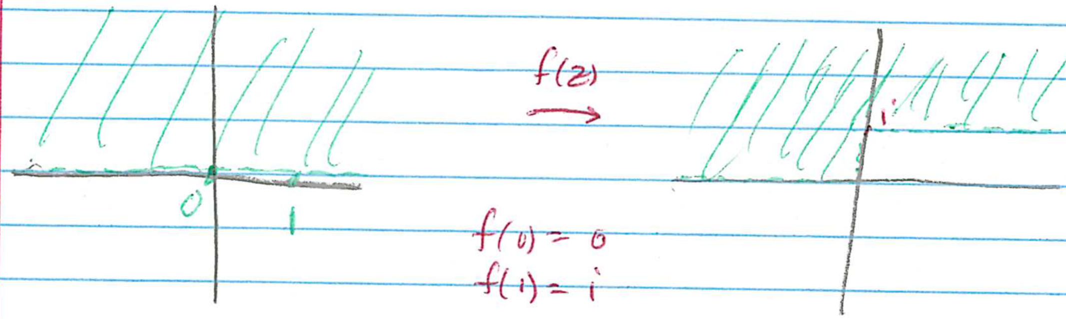
maps (conformal) on interior



#28 Variation $\quad$ Let $z_0 = 0$. $f(z) = a \int_0^z (w - z_1) \cdots (w - z_n) \, dw + b$



Example We wish to find a conformal bijection from



$f(0) = 0$
$f(1) = i$

Our map will be defined on the boundary but only conformal on the interior, $\mathrm{Im}(z) > 0$.

What does $\arg f'(z)$ on the real axis need to do?

$$\arg f' = \begin{cases} 0 & z < 0 \\ \pi/2 & 0 < z < 1 \\ 0 & z > 1 \end{cases}$$

Let's try $\arg f' = -\frac{1}{2} \arg z + \frac{1}{2} \arg (z-1)$. It works.

$$\arg z = \begin{cases} \pi & z < 0 \\ 0 & z > 0 \end{cases} \qquad \arg(z-1) = \begin{cases} \pi & z < 1 \\ 0 & z > 1 \end{cases}$$

Try $f'(z) = z^{-\frac{1}{2}} (z-1)^{\frac{1}{2}} = \left(\frac{z-1}{z}\right)^{1/2}$

This would give $f(z) = \int_0^z \sqrt{\frac{q-1}{q}}\, dq$

It will almost work. The bends will be correct and $f(0) = 0$. What is $f(1)$?

Well, $f(1) = \int_0^1 \sqrt{\frac{q-1}{q}}\, dq$. Let $q = \sin^2\theta$.
Then $dq = 2\sin\theta\cos\theta\, d\theta$,
and q goes from 0 to $\pi/2$.

and $\sqrt{\frac{q-1}{q}} = \sqrt{\frac{\sin^2\theta - 1}{\sin^2\theta}} = \sqrt{\frac{-\cos^2\theta}{\sin\theta}} = i\,\frac{\cos\theta}{\sin\theta}$.

Now,

$$f(1) = i\int_0^{\pi/2} \frac{\cos\theta}{\cancel{\sin\theta}}\, 2\,\cancel{\sin\theta}\cos\theta\, d\theta = 2i\int_0^{\pi/2}\cos^2\theta\, d\theta$$

$$= \cancel{2}i\int_0^{\pi/2} \frac{1+\cos(2\theta)}{\cancel{2}}\, d\theta = i\left(\theta + \tfrac{1}{2}\sin(2\theta)\right)\Big|_0^{\pi/2} = i\,\frac{\pi}{2}$$

So, with our initial choice for f we have $f(1) = i\frac{\pi}{2}$ not i.
Thus, we change our minds and let

$$f(z) = \frac{2}{\pi}\int_0^z \sqrt{\frac{q-1}{q}}\, dq$$

$$= \frac{2i}{\pi}\left(\arcsin\sqrt{z} + \sqrt{z(1-z)}\right).$$ ~~as shown on next page.~~

[In the substitution above, θ goes from 0 to $\arcsin\sqrt{z}$.

$\sin(2\arcsin\sqrt{z}) = 2\sin(\arcsin\sqrt{z})\cos(\arcsin\sqrt{z})$

$= 2\sqrt{z}\,\sqrt{1-z}$,]

Is this really onto? A version of the S-C Thm says yes,
but you can check $f(z = x+iy)$ for $y \to \infty$.

Plots of the images of horizontal lines under the tranform we found.

```
> f := z ->2*I/Pi*(arcsin(sqrt(z))+sqrt((z)*(1-z)));
```

$$f := z \rightarrow \frac{2\,I\left(\arcsin(\sqrt{z}) + \sqrt{z\,(1-z)}\right)}{\pi} \tag{1}$$

```
> f1:=complexplot( f(x+I/2),x=-5..10,color=red):  #image of the horizontal line z=x+I/2.
> f2:=complexplot( f(x+I)   ,x=-5..10,color=red):  #image of the horizontal line z=x+I.
> f3:=complexplot( f(x+2*I),x=-5..10,color=red):  #image of the horizontal line z=x+2I.
> f4:=complexplot( f(x+3*I),x=-5..10,color=red):  #image of the horizontal line z=x+3I.
> f5:=complexplot( f(x+5*I),x=-5..10,color=red):  #image of the horizontal line z=x+5I.
> b1:=line([-5,0], [0,0], color=black,thickness=3):
> b2:=line([0,0], [0,1], color=black,thickness=3):
> b3:=line([0,1], [5,1], color=black,thickness=3):
> display(f1,f2,f3,f4,f5,b1,b2,b3,view=[-5..5,0..4]);
```