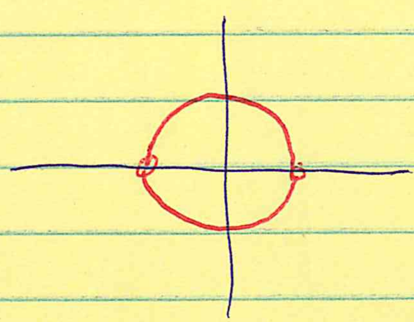
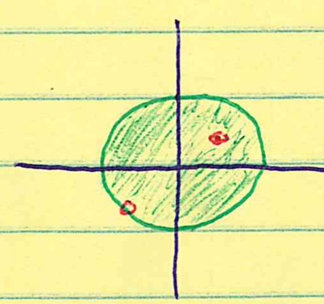
Cut Points

Def Let X be a connected space. A point $p \in X$ is a cut point if $X - \{p\}$ is not connected.

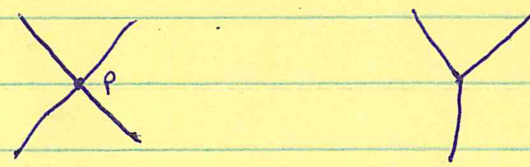
Ex We used this idea when we proved $\mathbb{R}$ is not homeo. to $\mathbb{R}^2$. The point $0 \in \mathbb{R}$ is a cut pt (every pt in $\mathbb{R}$ is a cut pt.), but no point in $\mathbb{R}^2$ is a cut pt.

Ex Let $S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\}$ and $D^2 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1\}$. Neither has a cut pt, but consider $p=(1,0)$ and $q=(-1,0)$. Let $X = S^1 - \{p,q\}$. Suppose $h: S^1 \to D^2$ is a homeo. Then $h: X \to D^2 - \{h(p), h(q)\}$ is a homeo. But X is disconnected while D^2 with any two pts removed is connected. (Check this.)



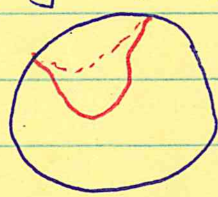


Ex Let X and Y represent the two spaces shown below, respectively.



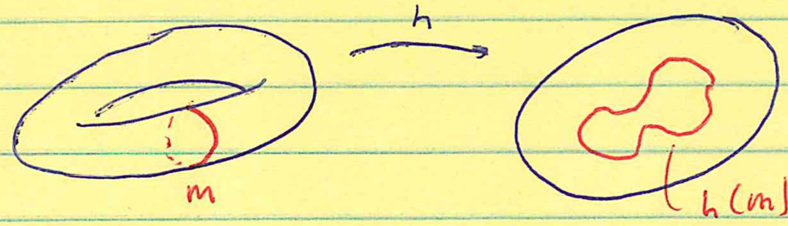
Suppose $h: X \to Y$ is a homeo. Then $h: X - \{p\} \to Y - h(p)$ is a homeo. But $X - \{p\}$ has four connected components, while $Y - \{h(p)\}$ has at most three components. This is impossible $h^{-1}: Y - \{h(p)\} \to X - \{p\}$ must take each component of $Y - \{h(p)\}$ into only one component of $X - \{p\}$. Hence h^{-1} cannot be onto.

The Jordan Curve Thm says that any simple closed curve J in $S^2 = \{(x,y,z) \mid x^2+y^2+z^2=1\}$ divides S^2 into two open disks. (see §63.) This means $S^2 - J$ is homeo to two disjoint open disks in $\mathbb{R}^2$. In particular $S^2 - J$ is disconnected; it has two components.



We will use this result to prove that S^2 is not homeo to T^2, the two dimensional torus.

Suppose $h: T^2 \to S^2$ is a homeo. Let m be the curve on T^2 shown below. Then $h(m)$ is a



simple closed curve in S^2. Now

$$h: T^2 - m \longrightarrow S^2 - h(m)$$

must be a homeo. But this cannot be since $T^2 - m$ is connected and $S^2 - h(m)$ is not by the Jordan Curve Thm.

Ex An n-dimensional manifold is a top. sp. where every pt has a nbhd homeo to $\mathbb{R}^n$. An n-dimensional manifold with boundary is a top. sp. where every pt either has a nbhd homeo to $\mathbb{R}^n$ or has a nbhd homeo to $\mathbb{R}^{n-1} \times [0, \infty)$. See §36. (We are leaving out some details here.)

S^2, T^2, $\{(x,y) \in \mathbb{R}^2 \mid x^2+y^2<1\}$ are 2-manifolds.

D^2 is a 2-manifold with bdy.

It is known that if M and N are n-manifolds, possibly with bdy, and $h: M \to N$ is a homeo, then $h: \partial M \to \partial N$ is a homeo.

It follows that D^2 is not homeo to S^2 since $\partial D^2 = S^1$ and $\partial S^2 = \phi$.

Let M be the Mobius band and let A be an annulus, both closed. Suppose $h: M \to A$ is a homeo. But this cannot be since ∂M has one component and ∂A has two!