Intro. §12 + some metric top.

Topology is the abstract study of continuity and related concepts like convergence and connectedness.

Def (Recall and see §20) A metric on a set X is a symetric function $d : X \times X \to [0,\infty)$ s.t. $d(p,q) = 0$ iff $p = q$ and $d(p,q) + d(q,r) \geq d(p,r)$. A pair (X, d) is called a metric space.

Def For $p \in X$, $\varepsilon > 0$, let $B_\varepsilon(p) = \{ q \in X \mid d(p,q) < \varepsilon \}$. It is called the open ball of radius ε and center p.

Ex See webpage for pictures of balls $B_1(0,0)$ in $\mathbb{R}^2$ for several different metrics.

Def Let $O \subset X$. Then O is open, wrt d, if it is the union of open balls or (equivalently) if $\forall p \in O$, $\exists \varepsilon > 0$ s.t. $B_\varepsilon(p) \subset O$.

Now we can defined what it means for a function $f : X \to Y$ to be continuous for given metrics on X and Y.

Def Let (X, d_X) and (Y, d_Y) be metric spaces and let $f: X \to Y$. Then f is continuous at $p \in X$ if

Ⓐ $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $f(B_\delta^X(p)) \subset B_\varepsilon^Y(f(p))$.

Or (equivalently)

Ⓑ $\forall$ open set V of Y containing $f(p)$, $\exists$ open set $U \subset X$ containing p s.t. $f(U) \subset V$.

For the proof that Ⓐ and Ⓑ are equivalent see Thm 21.1, pg 129.

Now, how to generalize these idea to sets where there is no metric? After many different approaches where tired the following definition was settled on in the early 1900s.

Def A collection $\mathcal{T}$ of subsets of X is a topology for X if

(i) $\phi, X \in \mathcal{T}$,
(ii) all unions of members of $\mathcal{T}$ are in $\mathcal{T}$,
(iii) all finite intersections of members of $\mathcal{T}$ are in $\mathcal{T}$.

A pair $(X, \mathcal{T})$ is called a topological space.

Fact Metric spaces are topological spaces where the open sets form a topology. (i) and (ii) are easy to check. (iii) takes a bit more effort. See pg 119-120 or most any analysis textbook.

Note: Infinite intersections of open sets can fail to be open.

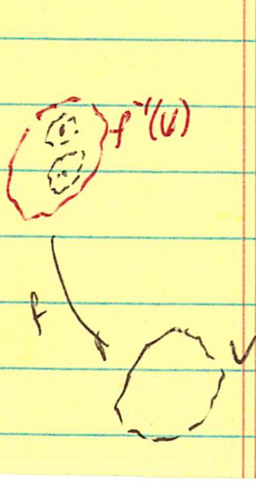
$$\bigcap_{n=1}^{\infty} \left(\frac{-1}{n}, \frac{1}{n} \right) = \{0\}.$$

Def We define continuity using (B).

Thm Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be top. spaces (not empty). Let $f: X \to Y$. Then f is continuous $\forall p \in X$ iff $\forall\, V \subset Y$ open, $f^{-1}(V)$ is open in X.

Pf Suppose f^{-1} maps open sets to open sets. Pick any $p \in X$ and let $V \subset Y$ be an open set containing $f(p)$. [Why does V exist?] Let $U = f^{-1}(V)$ and we are done.

Suppose f is cont. at all $p \in X$. Let $V \subset Y$ be open. If $f^{-1}(V) = \emptyset$ we are done. Assume $f^{-1}(V) \neq \emptyset$. (Thus $V \neq \emptyset$). We will show $f^{-1}(V)$ is open by showing it is a union of open sets. $\forall\, p \in f^{-1}(V)$ let U_p be open with $p \in U_p$ and $f(U_p) \subset V$. [$f(p) \in V$ and V is open so U_p exists.] Then



$$f^{-1}(V) = \bigcup_{p \in f^{-1}(V)} U_p. \quad \text{Thus, } f^{-1}(V) \text{ is open.} \quad \square$$

Examples Let $X \neq \phi$. Then $\{\phi, X\}$ is called the trivial top

The collection of all subsets of X is a top for X called the discrete top.

Let $f: X \to X$ be the identity function.
Assume X has at least two points.
What can you say about the continuity of f when

$$f: (X, \text{discrete}) \to (X, \text{trivial})$$
$$f: (X, \text{trivia}) \to (X, \text{discrete}) \quad ?$$

Let $f: X \to Y$ be any function. What can you say about the cont. of f when

$$f: (X, \text{any top.}) \to (Y, \text{trivial})$$
$$f: (X, \text{discrete}) \to (Y, \text{any top.})$$

Let $f: (X, \text{any top}) \to (Y, \text{my top})$ be a constant function: $\exists y_* \in Y$ s.t. $f(x) = y_*$ $\forall x \in X$. Show f is cont.

Def If $\mathcal{T}$ and $\mathcal{T}'$ are two topologies for X and $\mathcal{T} \subsetneq \mathcal{T}'$ (proper subset) we say $\mathcal{T}$ is coarser than $\mathcal{T}'$ or that $\mathcal{T}$ is finer than $\mathcal{T}$.

Examples on $\mathbb{R}$ Besides the trivial and discrete tops we have

$\mathcal{T}_u$ = unions of open intervals = usual top on $\mathbb{R}$ = ~~metric~~ standard metric top on $\mathbb{R}$.

$\mathcal{T}_p = \{ U \subset \mathbb{R} \mid \mathbb{R} - U \text{ is finite} \} \cup \{ \phi \}$
= finite complement topology

$\mathcal{T}_\ell$ = unions of subsets of the form $[a, b)$.

Notice $\mathcal{T}_{trivial} \subset \mathcal{T}_p \subset \mathcal{T}_u \subset \mathcal{T}_\ell \subset \mathcal{T}_{discrete}$

Q: For which topologies on $\mathbb{R}$ is there a metric?

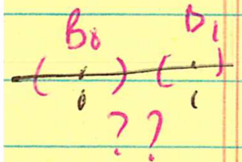
- Discrete? Yes, let $d(a,b) = \begin{cases} 0 & a = b \\ 1 & a \neq b. \end{cases}$

- Usual? Yes, $d(a,b) = |b-a|$.

- trivial? No. Pf: Suppose d is a metric on $\mathbb{R}$ that gives the trivial top. Let $\alpha = d(0,1) > 0$.
Let $I_0 = \{ x \in \mathbb{R} \mid d(0,x) < \alpha/2 \}$.
By def. I_0 is open. Thus $I_0 = \phi$ or $\mathbb{R}$.
$0 \in I_0$, thus $I_0 = \mathbb{R}$. But $1 \notin I_0$. Contradiction!
Thus, there is no metric on $\mathbb{R}$ that gives the trivial top.

— $\mathcal{T}_F$? No. Pf: Suppose d is a metric on $\mathbb{R}$ that gives $\mathcal{T}_F$ as only open sets. Let $\alpha = d(0,1) > 0$. Let

$$B_0 = B_{\alpha/3}(0) \quad \text{and} \quad B_1 = B_{\alpha/3}(1).$$



Since these must be open we can write

$$B_0 = \mathbb{R} - \{a_1, \dots a_n\} \qquad B_1 = \mathbb{R} - \{b_1, \dots b_m\}.$$

Since $\mathbb{R}$ is infinite $B_0 \cap B_1 \neq \emptyset$. Pick $x \in B_0 \cap B_1$.

Then $d(0,x) + d(x,1) \leq \frac{\alpha}{3} + \frac{\alpha}{3} = \frac{2\alpha}{3}$.

But, by the triangle ineq $d(0,x) + d(x,1) \geq d(0,1) = \alpha$.
Thus,

$$\frac{2}{3}\alpha \geq \alpha \Rightarrow \frac{2}{3} > 1 \Rightarrow 2 > 3.$$

Contradiction! $\square$

— $\mathcal{T}_L$? No. But we cannot prove this yet. See #6 on page 194.