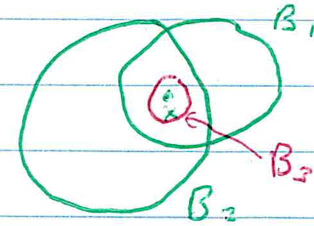
§13 Basis for a Topology

Def Let X be a set, $\mathcal{P}(X)$ denotes all subsets of X. Let $\mathcal{B} \subset \mathcal{P}(X)$ ~~and~~ that has the following two properties

(i) $\forall x \in X, \exists B \in \mathcal{B}$ s.t. $x \in B$.

(ii) $\forall B_1, B_2 \in \mathcal{B}$ and $\forall x \in B_1 \cap B_2$ $\exists B_3 \in \mathcal{B}$ s.t.

$$x \in B_3 \subset B_1 \cap B_2$$



Then we define $\mathcal{T} \overset{\subset \mathcal{P}(X)}{}$ as follows.

$$U \in \mathcal{T} \text{ iff } \forall x \in U \; \exists B \in \mathcal{B} \text{ with } x \in B \subset U.$$

Thm Then $\mathcal{T}$ is a top for X. (We say $\mathcal{B}$ is a basis for $\mathcal{T}$ or that $\mathcal{T}$ is generated by $\mathcal{B}$.)

Note: (i) is equivalent to requiring $\bigcup \mathcal{B} = X$.

Ex The open balls of a metric sp are a basis for the metric top.

Pf of Thm 1a. We claim $\emptyset \in \mathcal{T}$. For suppose not. then $\exists x \in \emptyset$ s.t. $\forall B \in \mathcal{B}$, $x \notin B$ or $B \not\subset \emptyset$. But no such x ~~sd~~ exists since there are no elements x is $\emptyset$. The $\emptyset \in \mathcal{T}$.

1b. By (i) $X \in \mathcal{T}$.

2. Let $\mathcal{U} \subset \mathcal{T}$. Let $U = \cup \mathcal{U}$. We claim $U \in \mathcal{T}$.
Let $x \in U$. Then $x \in U'$ for some $U' \in \mathcal{U}$.
Since $U' \in \mathcal{T}$ $\exists B \in \mathcal{B}$ s.t. $x \in B \subset U' \subset U$.
Thus $U \in \mathcal{T}$.

3. Let $U_1, \dots, U_n$ be in $\mathcal{T}$. Let $U = U_1 \cap U_2 \cap \dots \cap U_n$.
If $U = \emptyset$ we are done. Let $x \in U$. For $i = 1 \dots n$

$$\exists B_i \in \mathcal{B} \text{ s.t. } x \in B_i \subset U_i.$$

Let $C = \bigcap_{i=1}^{n} B_i$. By induction and (ii) $\exists B \in \mathcal{B}$ s.t.

$$x \in B \subset C \subset U.$$

Thus $U \in \mathcal{T}$ $\blacksquare$

Examples The collection of open intervals is a basis for the usual top. on $\mathbb{R}$.

The collection of one point sets in X is a basis for the descrete top. on X.

What would be a "good" basis for $\mathbb{R}_F$, that $\mathbb{R}$ with the finite complement top?

The usual top. on $\mathbb{R}^2$ that is generated by open balls is also generated by each of the following:

- The collection of open rectangles.
- the collection of open triangles.
- the collection of open balls with radius $\leq \frac{1}{2}$.
- The collection of open balls with rational center coordinates

~~Any to~~

Any top. is a basis for itself.

Let $\overline{\mathbb{R}}^2 = \mathbb{R}^2 \cup \{\infty\}$. Let $\mathcal{B} = \{$ open disks in $\mathbb{R}^2$ and the complements of closed bounded disks $\cup \{\infty\}\}$. The $\mathcal{B}$ is a basis for a top. on $\overline{\mathbb{R}}^2$ that is called the one-point compactification of $\mathbb{R}^2$. It turn out to be "homeomorphic" to $S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2=1\}$. See page 185.

Here are some useful fact whose proves you can read in the textbook. Assume X is a top sp.

Lemma 13.2 ~~Let~~ Suppose $\mathcal{C}$ is a collection of open sets of X s.t. for each open set U and each $x \in U$, $\exists\, C \in \mathcal{C}$ s.t. $x \in C \subset X$. Then $\mathcal{C}$ is a basis for the top. on X.

Lemma 13.3 Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for topologies $\mathcal{T}$ and $\mathcal{T}'$ resp. Then $\mathcal{T} \subset \mathcal{T}'$ iff

$\forall x \in X$ and $B \in \mathcal{B}$ with $x \in B$, $\exists\, B' \in \mathcal{B}'$ s.t. $x \in B' \subset B$.

* Also read Lemma 13.4, you will need it for the exercises.

Def Let X be a set and $\mathcal{S} \subset \mathcal{P}(X)$ s.t. $\bigcup \mathcal{S} = X$. Let $\mathcal{B}$ = all finite intersections of members of $\mathcal{S}$. Then $\mathcal{B}$ forms a basis for a topology $\mathcal{T}$ on X. (see bottom of pg 82 for proof.) Then $\mathcal{S}$ is called a <u>subbasis</u> for $\mathcal{T}$

Examples The open rays of $\mathbb{R}$ (sets of the form (a, ∞) and $(-\infty, a)$) is a subbasis for the usual top. on $\mathbb{R}$.

Let $\mathcal{S} = \{ \mathbb{R} - \{x\} \mid x \in \mathbb{R} \}$. Then $\mathcal{S}$ is a subbasis for the finite complete top. on $\mathbb{R}$.

In $\mathbb{R}^2$ let $\mathcal{S} = \{ U \times \mathbb{R} \mid U \text{ open interval in } \mathbb{R} \}$
$\cup \{ \mathbb{R} \times V \mid V \text{ open interval in } \mathbb{R} \}$.
Then $\mathcal{S}$ is a subbasis for the usual top on $\mathbb{R}^2$. $\mathcal{B}$ = open rectangles.