§14 The Order Topology

Def Recall that an order relation on a set X is a relation $<$ s.t.

1. $\forall x, y \in X$, if $x \neq y$ than exactly one of $x < y$ or $y < x$ holds.
2. $\forall x \in X$, $x < x$ is not true.
3. $\forall x, y, z \in X$, if $x < y$ and $y < z$ then $x < z$.

By convention $x > y$ means $y < x$ and $x \leq y$ means $x < y$ or $x = y$ holds. Recall the standard notation for intervals and rays.

$$\left.\begin{aligned}
[a,b] &= \{x \in X \mid a \leq x \leq b\} \\
(a,b] &= \{x \in X \mid a < x \leq b\} \\
[a,b) &= \{x \in X \mid a \leq x < b\} \\
(a,b) &= \{x \in X \mid a < x < b\}
\end{aligned}\right\} \text{Intervals}$$

$$\left.\begin{aligned}
[a,\infty) &= \{x \in X \mid a \leq x\} \\
(a,\infty) &= \{x \in X \mid a < x\} \\
(-\infty,b] &= \{x \in X \mid x \leq b\} \\
(-\infty,b) &= \{x \in X \mid x < b\}
\end{aligned}\right\} \text{rays (assuming } X \text{ has no largest or smallest element)}$$

Def Let $(X, <)$ be a set with an order relation and assume X has at least two members. If X has a largest element call it b_0. If X has a smallest element call it a_0.

$$\mathcal{B} = \{ (a,b),\ [a_0,b),\ (a,b_0],\ [a_0,b_0] \mid a,b \in X,\ a_0 < a < b < b_0 \}$$

Only include if they exist.

We claim $\mathcal{B}$ is a basis for a topology on X. It is called the Order topology.

Pf ① Let $x \in X$. If $x = a_0$, pick any $b \in X - \{a_0, b_0\}$. Then $[a_0, b) \in \mathcal{B}$. If $x = b_0$ pick any $a \in X - \{a_0, b_0\}$. Then $(a, b_0] \in \mathcal{B}$. If neither then

(i) $x \in [a_0, b_0]$ if both a_0 & b_0 exist,

(ii) $x \in [a_0, b)$ for some $b > x$ if b_0 does not exist but a_0 does

(iii) $x \in (a, b_0]$ for some $a < x$ if a_0 does not exist but b_0 does,

(iv) $x \in (a,b)$ for some $a < x$ and $b > x$.

② Assume neither a_0 or b_0 exists. Let $A = (a,b)$ and $B = (c,d)$. Either

(i) $A \cap B = \phi$

(ii) $A \cap B = (c,b)$

(iii) $A \cap B = A$

(iv) $A \cap B = B$.

Thus if $x \in A \cap B$, $\exists C \in \mathcal{B}$ s.t. $x \in C \subset A \cap B$.

You can check the other cases, i.e. a_0, b_0 both exist, a_0 only exists, and b_0 only exists. $\square$

<u>Ex</u> The standard top. on $\mathbb{R}$ is the same as the order top. This is also true for any interval or ray in $\mathbb{R}$ with the subsp. top.

The is not true for $\mathbb{R}^2$ since we do not have and ~~a~~ standard order relation. Let's put the dictionary order on $\mathbb{R}^2$ and see what the basis sets look like. Let $(x_1, y_1) < (x_2, y_2)$

$$\text{if} \quad x_1 < x_2 \quad \text{or} \left(x_1 = x_2 \text{ and } y_1 < y_2 \right).$$

~~(If neither hold we'd have $(x_2, y_2) < (x_1, y_2)$)~~

Both we show $((2,1), (3,0))$ and $((1,1), (1,3))$