§17

Closed Sets, Limit Points and Hausdroft Spaces!

Def Let X be a top. sp. Then $C \subset X$ is <u>closed</u> if $X - C$ is open.

Note ϕ and X are both open and closed. Such sets are called <u>clopen</u>.

Ex In the discrete top. all sets are clopen.

Thm (a) Finite unions of closed set are closed.
(b) Arbibrary intersections of closed sets are closed.

Pf (a) Let $C_1, \ldots, C_n$ be closed. Let $U_i = X - C_i$, $i = 1 \ldots n$. The U_i's are open. Now

$$X - \bigcup_{i=1}^{n} C_i = \bigcap_{i=1}^{n} (X - C_i) = \bigcap_{i=1}^{n} U_i$$

is open. Hence $\bigcup C_i$ is closed.

(b) Let C_α, $\alpha \in J$, be closed. Let $U_\alpha = X - C_\alpha$ $\forall \alpha \in J$. Then

$$X - \bigcap_{\alpha \in J} C_\alpha = \bigcup_{\alpha \in J} (X - C_\alpha) = \bigcup_{\alpha \in J} U_\alpha$$

is open. Hence $\bigcap C_\alpha$ is closed. $\boxtimes$

Fact $\bar{A}$ is closed and A° is open.

Ex $\bigcup_{x \in (0,1)} \{x\} = (0,1)$. Thus arbitrary unions of closed set need not be closed.

$\bigcup_{n=1}^{\infty} [\frac{1}{n}, 1] = (0, 1]$. Thus countable unions of closed set need not be closed.

Thm (17.2) Let Y be a subsp. of X. Then $A \subset Y$ is closed in Y iff $\exists$ $C \subset X$, closed, with $A = Y \cap C$.

Pf ($\Leftarrow$) Suppose $A = Y - C$, where C is closed in X. Then $X - C$ is open in X. Thus $Y \cap (X-C)$ is open in Y. But

$$Y \cap (X-C) = (X \cap Y) - (C \cap Y) = Y - A.$$

So $Y - A$ is open in Y. Hence A is closed in Y.

($\Rightarrow$) Suppose $A \subset Y$ is closed in Y. Then $Y - A$ is open in Y. Thus $\exists$ $U \subset X$, ~~closed~~ open with $U \cap Y = Y - A$. Let $C = X - U$; it is closed in X. But

$$C \cap Y = (X-U) \cap Y = (X \cap Y) - (U \cap Y) = Y - (Y - A) = A.$$

∎

Rmk In general A closed in Y does not imply A is closed in X. Ex: Let $A = (0,1]$, $Y = (0,2)$, $X = \mathbb{R}$. Then A is closed in the subsp. top on Y, but A is not closed in $X = \mathbb{R}$.

You will show in #2 that if A is closed in Y, and Y is closed in X, then A is closed in X.

Def Let A be a subset of a top. sp. X. The interior of A is the union of all open subsets of A. The closure of A is the intersection of all closed subsets of X that contain A. ~~It is apparent that~~

Notation $A^\circ = \text{Int}\, A$ = interior of A.
$\bar{A} = \text{Cl}\, A$ = closure of A.

Fact It is apparent that $\text{Int} A \subset A \subset \bar{A}$.

Ex For $A = [0,1)$, $A^\circ = (0,1)$, $\bar{A} = [0,1]$.
For $A = \{2\}$, $A^\circ = \phi$, $\bar{A} = \{2\}$.
For $A = (0,1) \cup (1,2)$, $\text{Int}\, \bar{A} = (0,2)$.
For $A = (0,1] \cup \{3\}$, $A^\circ = (0,1)$. $\bar{A} = [0,1] \cup \{3\}$.

Def The boundary or frontier of a set A is

$$Bd(A) = Fr(A) = \bar{A} \cap \overline{(X - A)}.$$

Ex For $A = (0,1] \cup \{3\}$, $\operatorname{Bd} A = \{0, 1, 3\}$.
For $\mathbb{Q}$ as a subset of $\mathbb{R}$, $\operatorname{Bd} \mathbb{Q} = \overline{\mathbb{Q}} = \mathbb{R}$ and $\operatorname{Int} \mathbb{Q} = \emptyset$.

Def A neighborhood (nbhd) of a pt x in a top sp X is any open subset of X that contains x.

Thm (a) $x \in \overline{A} \iff$ Every nbhd of x intersects A.
(b) $x \in A^{\circ} \iff \exists$ a nbhd of x contained in A.

pf (a) ($\Rightarrow$) Suppose $x \in \overline{A}$ and $\exists$ nbhd of x, U, s.t. $U \cap A = \emptyset$. Let $C = X - U$. Then $A \subset C$, a closed set that does contain x. Contradiction.

($\Leftarrow$) Suppose $x \notin \overline{A}$. Then $\exists$ a closed set C s.t. $A \subset C$ and $x \notin C$. Let $U = X - C$. Then U is a nbhd of x that does not intersect A. The conclusion follows.

(b) ($\Rightarrow$) Suppose $x \in A^{\circ}$. Then $\exists$ open set U s.t. $x \in U \subset A$.

($\Leftarrow$) Let U be a nbhd of x contained in A. Then $x \in A^{\circ}$. $\boxtimes$

Many Facts See the exercises.

Limit Points

Def Let $A \subset X$, a top. sp. Then $x \in X$ is a limit point of A if every nbhd of x meets $A - \{x\}$.

Ex Let $A = \bigcup_{n=1}^{\infty} \{\frac{1}{n}\} \subset \mathbb{R}$. Then 0 is the only limit pt of A.

Ex Let $A = (0,2] \cup \{3\}$. The limit pts of A are $[0,2]$.

Notation Let $A' = \{$ limit points of $A \}$.

Thm (17.6) $\bar{A} = A \cup A'$.

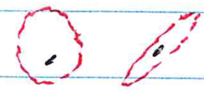
pf See textbook.

Fact A is closed iff $A' \subset A$.

Note A limit point of A is not the same as a point that is the limit of seq in A. For $A = \{1\} \cup \{2\} \cup \{3\}$, there are no limit points, $A' = \emptyset$. Thus, the image of a constant function has no limit points even if the domain of the function has limit pts.

Hausdorff Spaces

Def A top sp X is **Hausdorff** if $\forall x, y \in X, x \neq y$, $\exists\ U, V$ open with $x \in U$, $y \in V$ and $U \cap V = \phi$.



Thm One pt sets are closed in a Hausdorff Sp.

Pf Let $x_0 \in X$. We will show that $\overline{\{x_0\}} = \{x_0\}$. Let $x \in X - \{x_0\}$. Let U, V open separate x and x_0. ($x \in U$, $x_0 \in V$, $U \cap V = \phi$.) Then $U \cap \{x_0\} = \phi$ which implies $x \notin \overline{\{x_0\}}$. Thus, the only member of $\overline{\{x_0\}}$ is x_0.

(Thus, X-{x_o} is open.)

Cor By induction finite subsets of H. sp.'s are closed.

Thm Let X be a top. sp. in which finite sets are closed. Let $A \subset X$. Then $x \in A'$ iff every nbhd of x contains infinitely many pts of A.

Pf The "if" direction is obvious. Let $x \in A'$ and suppose some nbhd U of x is such that

$$(U \cap A) - \{x\} = \{x_1, x_2, \dots x_m\}.$$

Then $X - \{x_1, \dots, x_m\}$ is open. Let $U' = U \cap (X - \{x_1, \dots, x_m\})$ $= U - \{x_1, \dots, x_m\}$. Then U' is an open nbhd of x that misses A, $U' \cap A = \phi$.

(Note: U' does not mean the limit pts of U.)

Def Let $\{x_n\}_{n=1}^{\infty}$ be a seq in a top. sp. X. Let $x \in X$. We say $x_n \to x$ if $\forall$ nbhd U of x, $\exists N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow x_n \in U$.

Thm In a Hausdorff sp limits are unique.

Pf Let X be H. with $x_n \to x$ and $x_n \to y$, with $x \neq y$.
Let U_x, U_y sep. x and y.
Let $N_x \in \mathbb{N}$ be s.t. $n \geq N_x \Rightarrow x_n \in U_x$.
Let $N_y \in \mathbb{N}$ be s.t. $n \geq N_y \Rightarrow x_n \in V_x$.
Let $N = \max\{N_x, N_y\}$. Let $n \geq N$.
Then $x_n \in U_x$ and $x_n \in V_x$. This is impossible. $\boxed{\star}$

Ex Consider the finite comp. top. on $\mathbb{R}$. Then $\{n^2\}_{n=1}^{\infty}$ converges to 17. Pf: Every nbhd U of 17 is the complement of a finite set, say $\{x_1, x_2, \dots x_n\}$. Let $x^* = \max\{x_1, \dots, x_n\}$. If $n^2 > x^*$ then $n^2 \in U$. $\square$

* You will show that subsp's of H. sp's are H.

* You will show that the prod of two H. sp's is H.