§18 Continuous Functions

Def Let $f: X \to Y$. Then f is continuous if the inverse images of open sets are open.

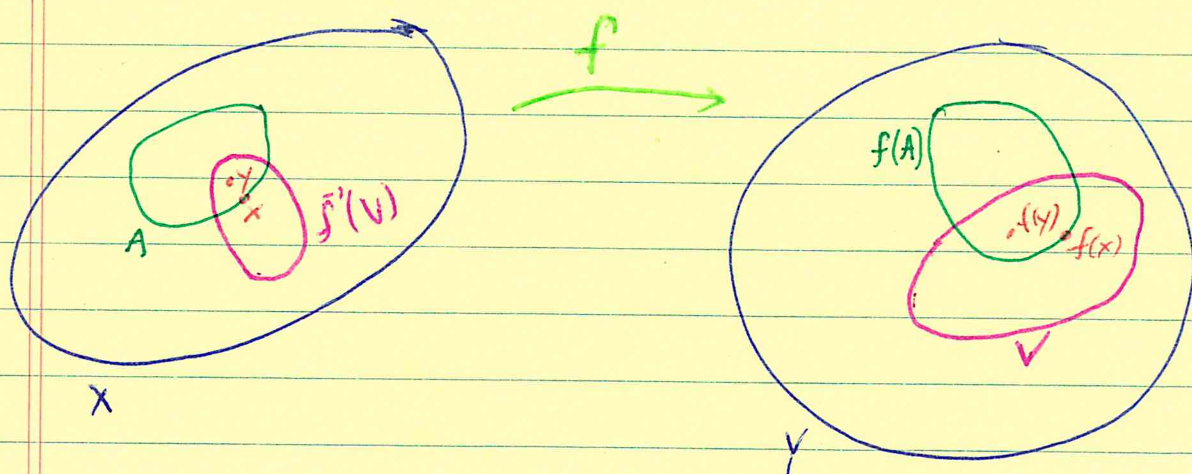
Thm 18.1 Let $f: X \to Y$. The following are equivalent.

1. f is cont.
2. $\forall A \subset X$, $f(\overline{A}) \subset \overline{f(A)}$.
3. Inverse images of closed set are closed.
4. $\forall x \in X$ and nbhd V of $f(x)$, $\exists\, U \subset X$, an open nbhd of x, s.t. $f(U)$

Note We proved ① $\Leftrightarrow$ ④ earlier.

Pf ① $\Rightarrow$ ② Let $A \subset X$ and $x \in \overline{A}$. We will show $f(x) \in \overline{f(A)}$. Let V be a nbhd of $f(x)$. Then $f^{-1}(V)$ is a nbhd of x. Thus $f^{-1}(V) \cap A \neq \emptyset$.

Let $y \in f^{-1}(V) \cap A$. Then $f(y) \in V \cap f(A)$. Thus, $V \cap f(A) \neq \emptyset$ and hence that $f(x) \in \overline{f(A)}$. $\blacksquare$



Pf ② ⇒ ③ Let $C \subset Y$ be closed. Let $A = f^{-1}(C)$. We will show that $A = \bar{A}$ and hence that A is closed.

Let $x \in \bar{A}$. Then

$$f(x) \in f(\bar{A}) \subset \overline{f(A)} \subset \overline{C} = C.$$

Since $f(x) \in C$, $x \in f^{-1}(C) = A$. Thus $\bar{A} \subseteq A$. Since we know $A \subset \bar{A}$, we have $A = \bar{A}$ and A is closed.

Pf ③ ⇒ ① Let $V \subset Y$ be open. Then $C = Y - V$ is closed. Thus,

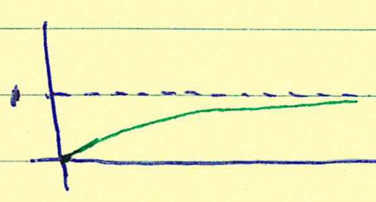
$$f^{-1}(C) = f^{-1}(Y) - f^{-1}(V) = X - f^{-1}(V).$$

Since $f^{-1}(C)$ is given to be closed, $f^{-1}(V)$ is open.

Note You can prove ① ⇒ ③ directly, that is without going through ②.

Ex We show it can happen that $f(\bar{A}) \neq \overline{f(A)}$, with f cont. Let $f: [0,\infty) \to [0,\infty)$ be given by $f(x) = x/(x+1)$. Let $A = [0,\infty)$. Then $\bar{A} = [0,\infty)$, but

$$f(\bar{A}) = [0,1) \neq \overline{f(A)} = \overline{[0,1)} = [0,1].$$



Thm 18.2 This is just a list of standard facts about which func's are cont. The proofs are in the textbook, but you should be able to do any of these on your own.

(a) Constant functions are cont.

(b) Inclusion maps are cont. [For $A \subset X$, the map $j: A \to X$ given by $j(a) = a$ is called an inclusion map. The identity map is a special case.]

(c) Suppose $f: X \to Y$ and $g: Y \to Z$ are cont. Then $g \circ f: X \to Z$ is cont.

(d) If $f: X \to Y$ is cont and $A \subset X$, then $f|A: A \to Y$ is cont.

(e) If $f: X \to Y$ is cont and $Y \subset Z$, then $f: X \to Z$ is cont.

(f) Let $X = \bigcup U_\alpha$, each U_α is open. Suppose $f: X \to Y$ and each $f|U_\alpha$ is cont. Then $f: X \to Y$ is cont.

See textbook for proofs. They are easy.

Thm (The Pasting Lemma) This is elementary but useful.
Let $X = A \cup B$, with A and B closed. Let $f : A \to Y$ and $g : B \to Y$ be cont. and suppose $f(x) = g(x)$ $\forall x \in A \cap B$.
Define $h : X \to Y$ by

$$h(x) = \begin{cases} f(x) & \text{for } x \in A, \\ g(x) & \text{for } x \in B. \end{cases}$$

Then h is cont.

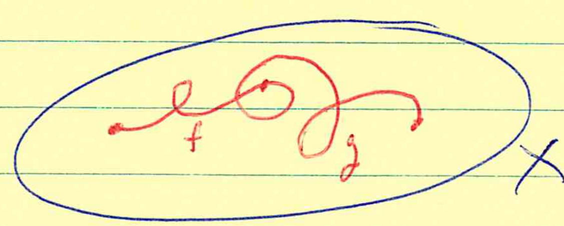
Pf Let $C \subset Y$ be closed. Now $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$. Since $f^{-1}(C)$ is closed in A and A is closed in X, $f^{-1}(C)$ is closed in X. Likewise $g^{-1}(C)$ is closed in X. Hence $h^{-1}(C)$ is closed and thus h is cont.

Note This holds for $X =$ a finite union of closed sets.
Compare with (f) of Thm 18.2.

Ex A *path* in a top sp X is a cont map $f : [0,1] \to X$.
Suppose f and g are paths in X and $f(1) = g(0)$.
Define

$$h = \begin{cases} f(2t) & t \in [0, \tfrac{1}{2}] \\ g(2t-1) & t \in [\tfrac{1}{2}, 1]. \end{cases}$$

Now $f(2t)$ and $g(2t-1)$ are cont. by 18.2 (c).
They are equal at $t = 1/2$. By the Pasting Lemma $h : [0,1] \to X$ is a path. The notation is $h = f * g$.



Ex Let $X = [0,2]$, $A = [0,1]$, $B = (1,2]$. Let $f: A \to \mathbb{R}$ be the constant 1, and let $g: B \to \mathbb{R}$ be 2. f and g argee on $A \cap B$ since $A \cap B = \emptyset$. But

$$h(x) = \begin{cases} 1 & x \in [0,1] \\ 2 & x \in (1,2] \end{cases}$$

is not cont. Thus the condition that A & B both be closed or both be open is necessary.

Thm 18.4 Let $f: A \to X \times Y$ be given by $f(a) = (f_1(a), f_2(a))$ where $f_1 : A \to X$ and $f_2 : A \to Y$. Then f is cont. iff f_1 and f_2 are cont.

Pf Suppose f is cont. Let $\pi_1 : X \times Y \to X$ be given by $\pi_1(x,y) = x$ and $\pi_2 : X \times Y \to Y$ be $\pi_2(x,y) = y$. (These are called projection maps. You can check that they are cont.)

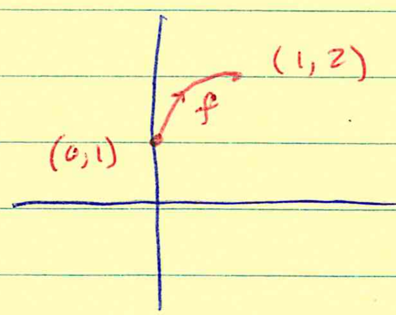
Now for $i = 1, 2$ we have $f_i(a) = \pi_i(f(a))$. These are each compositions of cont. functions and hence are cont.

Now suppose f_1 and f_2 are cont. Let $U \times V \subset X \times Y$ be a basic open set. Then

$$f^{-1}(U \times V) = f_1^{-1}(U) \cap f_2^{-1}(V)$$

is open. Hence f is cont.

Ex Let $f: [0,1] \to \mathbb{R}^2$ be $f(t) = (t^3, t^2+1)$. Since t^3 and t^2+1 are cont, f is cont and hence f is a path in $\mathbb{R}^2$.



#11 Let $F : X \times Y \to Z$ be cont. Show that F is cont. in each variable, that is $h = F|X \times y_0$ and $k = F|x_0 \times Y$ are cont $\forall\ x_0 \in X,\ y_0 \in Y$.

Pf Let $U \subset Z$ be open. Then $k^{-1}(U) = F^{-1}(U) \cap (x_0 \times Y)$ is open in the subspace top of $x_0 \times Y$. Thus k is cont. Likewise for h. Or cite Thm 18.2 (d).

Ex $F(x,y) = x^2 y$ is cont. and so are $h(x) = x^2 y_0$ and $k(y) = x_0^2 y$

#12 This gives an example showing the converse of #11 is false. Let

$$F(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Show that F is cont in each variable, but that F is not cont.

Pf Let $k : Y \to Z$ be $k(y) = F(x_0, y)$. If $x_0 \neq 0$, then

$$k(y) = \begin{cases} \frac{x_0 y}{x_0^2+y^2} & y \neq 0, \\ 0 & y = 0. \end{cases}$$

Then k is cont for $y \neq 0$. Since $\lim_{y \to 0} k(y) = 0 = k(0)$, we see that k is cont everywhere. If Xo is 0, then k(y)=0 always.

The proof that $h : X \to Z$ is cont. is similar.

Here are three ways you can show that $F : \mathbb{R}^2 \to \mathbb{R}$ is not cont. at $(0,0)$.

(1) Show that $\lim_{(x,y)\to(0,0)} F(x,y)$ does not exist by showing that $\lim_{x\to 0} F(x,0) \neq \lim_{x\to 0} F(x,x)$. You did this in Calc III.

(2) You can do an ε-δ style proof. Let $\varepsilon = 1/10$. Show that for any $\delta > 0$ $F(B((0,0),\delta))$ contains points outside $(-\frac{1}{10}, \frac{1}{10})$. This shows $\lim_{(x,y)\to(0,0)} F(x,y)$ cannot be $0 = F(0,0)$.

(3) Find an open set $U \subset \mathbb{R}$ s.t. $F^{-1}(U)$ is not open.

Hint: Write F(x,y) in polar coordinates.
U can be a small open set containing 0. (-1/4,1/4) works out well.

Homeomorphisms

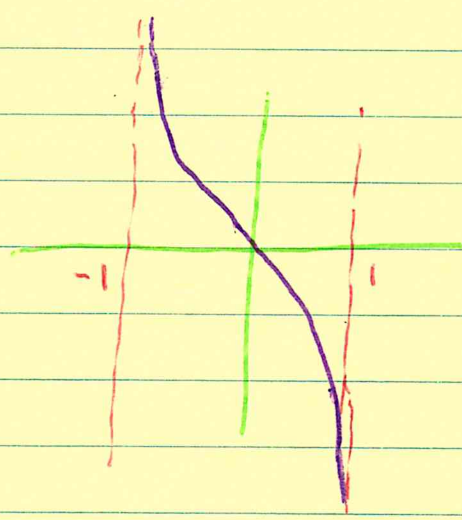
Def Let X and Y be top. spaces and $f: X \to Y$. If f is one-to-one, onto, continuous with cont. inverse then we say that f is a homeomorphism and that X and Y are homeomorphic or topologically equivalent.

Def A property of a top. sp. that is preserved under homeomorphisms is said to be a topological property.

> The classification of top. sp's up to homeomorphisms is the major project of topology.

Ex $(0,1)$, $(-7,18)$, $(5,\infty)$ and $\mathbb{R}$ are all homeomorphic. Thus, boundedness is not a top. property. Find equations for these.

Let $h: (-1,1) \to \mathbb{R}$ be given by $h(x) = \dfrac{x}{x^2-1}$.
It is a homeo. Find a formula for h^{-1}.



Ex Any two closed bounded intervals in $\mathbb{R}$ are homeo. But $[0,1]$ and $[0,\infty)$ are not. How to prove this? This will come later.

Q Let $D = \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2 \leq 1\}$, $S = [0,1]\times[0,1]$ and $A = \{(x,y) \mid 1 \leq x^2+y^2 \leq 4\}$. Which of these do you think are homeo?

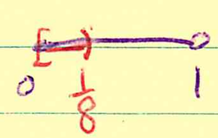
Note Let $f: X \to Y$. This induces a function $\hat{f}: \mathcal{T}_X \to \mathcal{T}_Y$ given by $\hat{f}(U) = \{f(x) \mid x \in U\}$ if f is a homeo. In fact $\hat{f}$ is a bijection. The "topological structures" of X and Y are "the same".

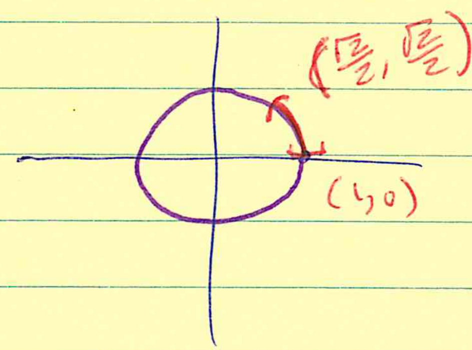
Ex It can happen that $f: X \to Y$ is cont, 1-to-1 and onto, without f^{-1} being cont. Here is an example.

Let $S^1 = \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2=1\}$. Let $f: [0,1) \to S^1$ be given by

$$f(t) = (\cos(2\pi t), \sin(2\pi t)).$$

Then f is a cont, bijection. But $f([0,\frac{1}{8}))$ is not open in S^1 even though $[0,\frac{1}{8})$ is open in $[0,1)$ (subsp. top.).





See Thm 26.6 on pg 167 for conditions where cont. bij $\Rightarrow$ homeo.

Q Is being Hausdorff a top. property?

A Yes!. Let $f: X \to Y$ be a homeo. and suppose X is Hausdorff. Let $p, q \in Y$, $p \neq q$. Let $a = f^{-1}(p)$ and $b = f^{-1}(q)$. We know $a \neq b$. Let U, V separate a and b, that is, both are open and

$$a \in U, \; b \in V, \; U \cap V = \emptyset.$$

Since f^{-1} is cont, $f(U)$ and $f(V)$ are open in Y. Since f is one-to-one $f(U) \cap f(V) = \emptyset$. Clearly $p \in f(U)$ and $q \in f(V)$. Thus Y is Hausdorff $\boxed{+}$