§19 The Product Topology

Def Let $\{X_\alpha\}_{\alpha \in J}$ be a family of disjoint sets indexed by J. Then

$$\prod_{\alpha \in J} X_\alpha = \{ (x_\alpha)_{\alpha \in J} \mid x_\alpha \in X_\alpha \},$$

where $(x_\alpha)_{\alpha \in J}$ is called a J-tuple. You can think of $(x_\alpha)_{\alpha \in J}$ as a function $J \to \bigcup_{\alpha \in J} X_\alpha$ with $J(\alpha) \in X_\alpha$.

Def Let $X = \prod X_\alpha$. Let $\mathcal{B} = \{ U = \prod U_\alpha \mid U_\alpha \subset X_\alpha \text{ is open} \}$. Then $\mathcal{B}$ generates the box topology on X.

Rmk This topology has too many open sets!

Ex 1 Let $f : \mathbb{R} \to \mathbb{R}^\omega \equiv \prod_{n \in \mathbb{N}} \mathbb{R}$, be given by

$$f(t) = (t, t, t, \dots)$$

Let $B = (-1,1) \times (-\frac{1}{2}, \frac{1}{2}) \times (\frac{-1}{3}, \frac{1}{3}) \times \cdots = \prod_{n=1}^{\infty} (\frac{-1}{n}, \frac{1}{n})$.

But $f^{-1}(B) = \{0\}$ which is not open. Thus f is not continuous even though ~~it~~ it is continuous in each variable.

Ex 2 Let $\mathbf{S} = (S_i)_{i=1}^{\infty}$ where $S_1 = (1, 0, 0, \ldots)$, $S_2 = (1, 1, 0, \ldots)$
$S_3 = (1, 1, 1, 0, 0, \ldots)$ .....

Then we'd expect $S_i \to (1, 1, 1, \ldots)$.
But it does not! Let $B = (\frac{1}{2}, \frac{3}{2}) \times (\frac{1}{2}, \frac{3}{2}) \times \cdots = \prod_{i=1}^{\infty} (\frac{1}{2}, \frac{3}{2})$.
No S_i is in B.

We need a better top. to do reasonable math with.

Def Let $X = \prod_{\alpha \in J} X_\alpha$. Let $\mathcal{B} = \{ \prod U_\alpha \mid U_\alpha \subset X_\alpha$ is open and all but finitely many $U_\alpha = X_\alpha \}$
Then $\mathcal{B}$ generates a top. on X called the ~~prod. top~~ product top.

Rmk Let $\mathcal{S} = \{ \pi_\alpha^{-1}(U_\alpha) \mid U_\alpha$ open in X_α, $\alpha \in J \}$.

Then $\mathcal{S}$ is a subbasis for the prod. top whether J is finite or infinite.

Ex. $f : t \to (t, t, t, \ldots)$ is now cont. wrt prod top.

$S_i \to (1, 1, 1, \ldots)$ in prod. top.

Thm (19.6) Let $f: A \to \prod_{\alpha \in J} X_\alpha$ be given by $f(a) = (f_\alpha(a))_{\alpha \in J}$, where $f_\alpha : A \to X_\alpha$. Then f is cont. iff each f_α is cont, if we use the prod. top. on $\prod X_\alpha$.

Pf See textbook.

Thm (19.4) If each X_α is Hausdorff, then $\prod X_\alpha$ is H. in either the prod or box top's.

Pf It is enough to show this for the prod. top. Why?

Let $x = (x_\alpha)_J$, $y = (y_\alpha)_J$ be in $\prod_J X_\alpha$, $x \neq y$. Then for at least one $\alpha' \in J$ $x_{\alpha'} \neq y_{\alpha'}$.

Let $U_{\alpha'}, V_{\alpha'}$ be disjoint nbhds of $x_{\alpha'}$ and $y_{\alpha'}$ in $X_{\alpha'}$. Let

$$U = \left(\prod_{\alpha \in J - \{\alpha'\}} X_\alpha\right) \times U_{\alpha'} \quad \text{and} \quad V = \left(\prod_{\alpha \in J - \{\alpha'\}} X_\alpha\right) \times V_{\alpha'}.$$

Then $x \in U$, $y \in V$, $U \cap V = \emptyset$ and both are open. □

Thm 19.5 Let $A_\alpha \subset X_\alpha$, $\forall \alpha \in J$. Then $\overline{\prod A_\alpha} = \prod \bar{A}_\alpha$ in both box and prod. top's.

Pf It is enough to show this for the box top.

Let $(x_\alpha) \in \prod \bar{A}_\alpha$. Let V be an open nbhd of (x_α). $\exists$ a basis member $U = \prod U_\alpha$ s.t. $(x_\alpha) \in U \subset V$. Clearly each $x_\alpha \in U_\alpha$. Since $x_\alpha \in \bar{A}_\alpha$ we know $U_\alpha \cap A_\alpha \neq \phi$. Thus $U \cap \prod A_\alpha \neq \phi$ and so $V \cap \prod A_\alpha \neq \phi$. Thus $(x_\alpha) \in \overline{\prod A_\alpha}$. Hence $\prod \bar{A}_\alpha \subset \overline{\prod A_\alpha}$.

Let $(x_\alpha) \in \overline{\prod A_\alpha}$. Let $x_\alpha \in U_\alpha$ where U_α is open in X_α, for each α, Then $\prod U_\alpha$ is an open nbhd of (x_α). Therefore $\prod U_\alpha \cap \prod A_\alpha \neq \phi$. Hence each $U_\alpha \cap A_\alpha \neq \phi$. Thus $x_\alpha \in \prod \bar{A}_\alpha$, and so $\overline{\prod A_\alpha} \subset \prod \bar{A}_\alpha$.

Thus $\prod \bar{A}_\alpha = \overline{\prod A_\alpha}$. $\square$

Why does this work for the prod. top?