§22 The Quotient Topology

Informal Examples

gluing

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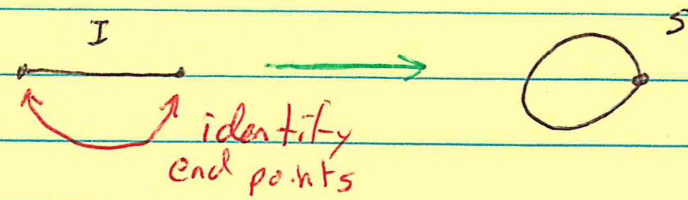

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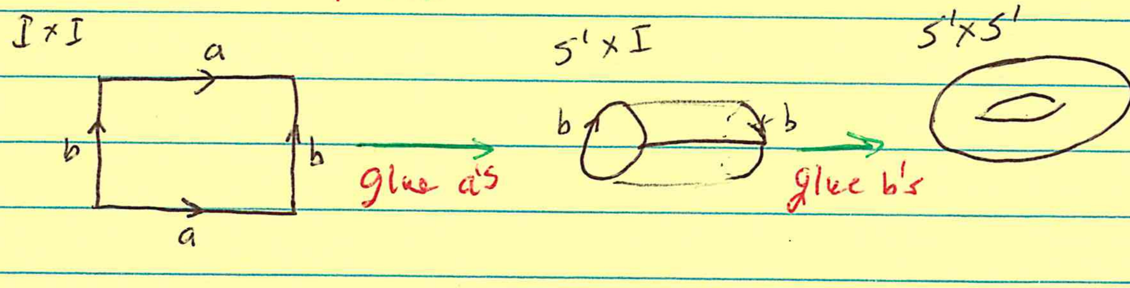

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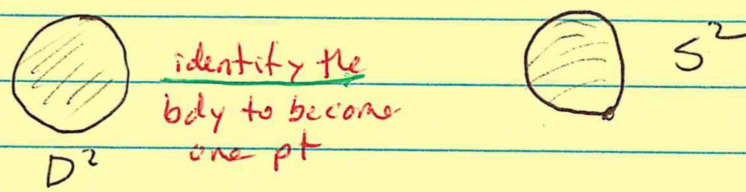

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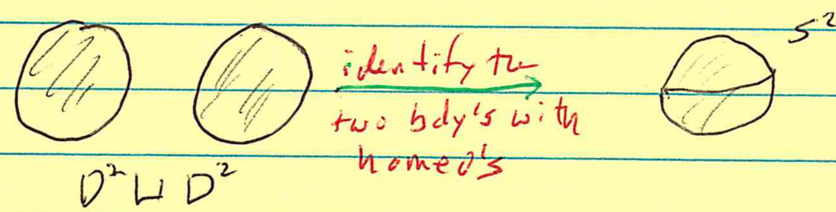

foliations

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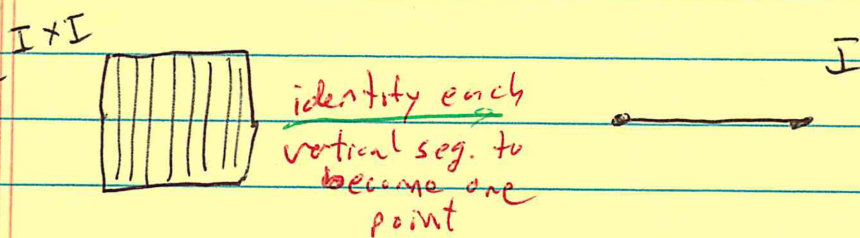

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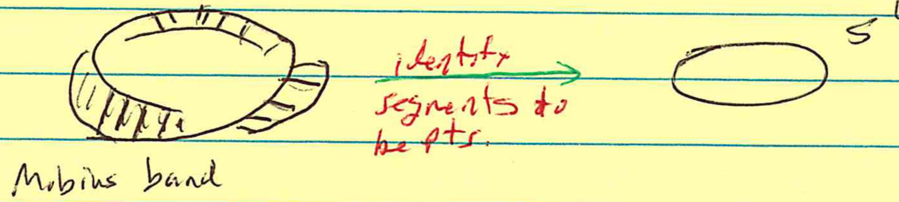


group actions

7,8 $\mathbb{R}/\mathbb{Z} \xrightarrow{n=n+1} S^1 \quad \mathbb{R}^2/\mathbb{Z}^2 \xrightarrow[m=m+1]{n=n+1} T^2 = S^1 \times S^1$

9 $S^2 \xrightarrow[\text{points}]{\text{identify opposite}} P^2$ (projective plane)

Def Let X and Y be top. sp's. Let $p: X \to Y$ be onto. Then p is called a quotient map provided $U \subset Y$ is open iff $p^{-1}(U)$ is open in X. [Clearly q-maps are cont., sometimes they are called strongly cont.]

The def is equivalent to $C \subset Y$ is closed iff $p^{-1}(C) \subset X$ is closed, as you can check.

Ex A homeo. is trivially a q-map.
Projection maps are q-maps.

The map $\mathbb{R} \to \mathbb{R}$ given by $x \to x^2$ is not a q-map b/c it is not onto. But if we take this map to be from $\mathbb{R} \to [0, \infty)$ it is a q-map.
Check this.

Def A map $f: X \to Y$ is open if it takes open sets to open sets and is closed if it takes closed sets to closed sets.

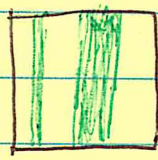
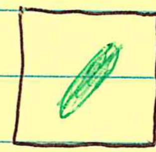
Fact Cont. onto maps that are open or closed are q-maps. But $\exists$ q-maps that are neither open nor closed. (See Exercise #3; it is assigned.)

Def Let $p : X \to Y$ be onto. Then $A \subset X$ is **saturated**, wrt p, if $A = p^{-1}(p(A))$.

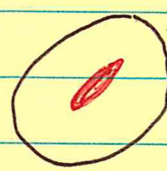
Ex Let $f(x) = x^2$ be a map $\mathbb{R} \to [0,\infty)$. Then $\{0\}$, $\{-1,1\}$, $(-1,1)$, $[-3,3]$ and $\mathbb{R}$ are saturated, while $[0,1]$, $[-1,1)$, and $\{5\}$ are not saturated.

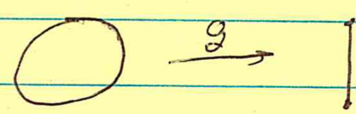
For a homeo. all sets are saturated.

Let $p : [0,1]^2 \to [0,1]$ be $p(x,y) = x$. Then

 is sat., while  is not.

Let $g : S^2 \to [-1,1]$ be given by $g(x,y,z) = z$. Here $S^2 = \{(x,y,z) \mid x^2+y^2+z^2=1\}$. Then

 is sat., while  is not.



Fact Let $p : X \to Y$ be onto and cont. It is a q. map iff it maps sat. open sets to open sets, iff it maps sat. closed sets to closed sets. Just check the definitions.

Def Let X be a top. sp. Let $p: X \to Y$ be onto. If $\mathcal{T}_X$ is the top. on X, let $\mathcal{T}_Y = \{ U \subset Y \mid p^{-1}(U) \in \mathcal{T}_X \}$. Now p is a q-map and $\mathcal{T}_Y$ is the unique top. on Y for which this is true. This top. is called the *quotient top.*

We are now going to define a large class of q-maps that are based on equivalence relations.

Def Let X be a top. sp. and let $\sim$ be an eq. rel. on X. Let X^* denote the eq. classes of $\sim$. Let

$$p: X \to X^*$$

be the map that takes each point to its eq. class,

$$p(x) = [x] \in X^*.$$

Given X^* the q-top. Then X^* is the quotient space of X mod $\sim$. We may write

$$X^* = X/\sim.$$

In fact all q-maps, $p: X \to Y$, can be thought of this way: let $a \sim b$ iff $p(a) = p(b)$. Then Y is homeo. to $X/\sim$.

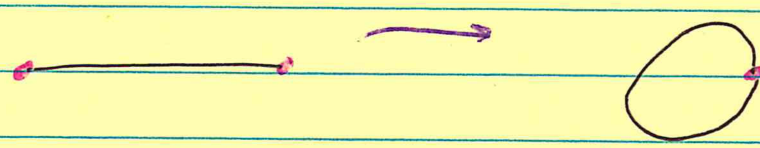
Ex For $I=[0,1]$ define $\sim$ by $x \sim y$ if one of the following hold $x=y$; $x=0, y=1$; $x=1, y=0$. Then

$$I^* = \{\{0,1\}\} \cup \{x \mid 0<x<1\}.$$

The q-top for I^* is generated by sets of the form (a,b) for $0<a<b<1$, $\{0,1\} \cup (0,a) \cup (b,1)$ where $0<a<b<1$.

We can map I^* homeomorphically to S^1 by

$h: I^* \to S^1$ where $h(t) = (\cos 2\pi t, \sin 2\pi t)$ for $0<t<1$ and $h(\{0,1\}) = (1,0)$.

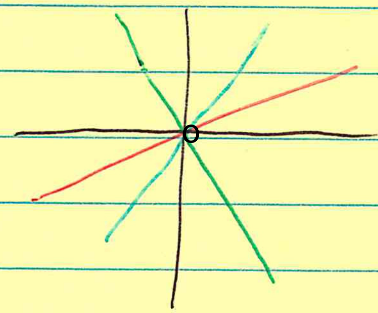


You can check that this is a homeo.

Ex On $\mathbb{R}$ let $x \sim y$ mean $x = y+n$ for some $n \in \mathbb{Z}$.

Thinking of $\mathbb{R}$ as a group under addition and $\mathbb{Z}$ as a subgroup $\mathbb{R}/\sim \; = \mathbb{R}/\mathbb{Z}$.
Let $h(t) = \{t+n \mid n \in \mathbb{Z}\}$. Then $\mathbb{R}/\mathbb{Z}$ is homeo to S^1.
The proof is on page 337.

Ex On $\mathbb{R}^2 -$ origin let $x \sim y$ iff they are on the same line going through the origin

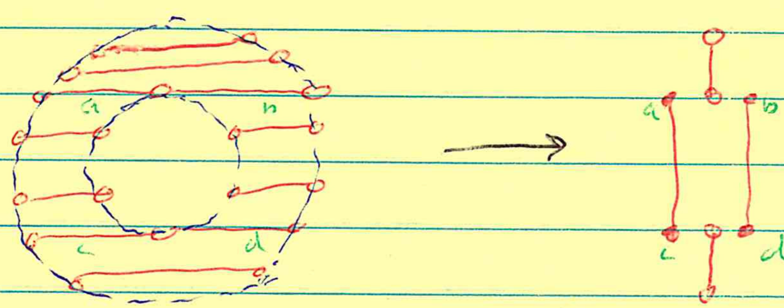


$$\mathbb{R}^2 - \text{origin}/\sim \;=\; \{\, \{(x,y) \mid y = mx\} \mid m \in \mathbb{R} \,\} \cup \{\, (0,y) \mid y \in \mathbb{R} - 0 \,\}.$$

It can be shown that $\mathbb{R}^2 - \text{origin}/\sim$, with the q-top, is homeomorphic to S^1.

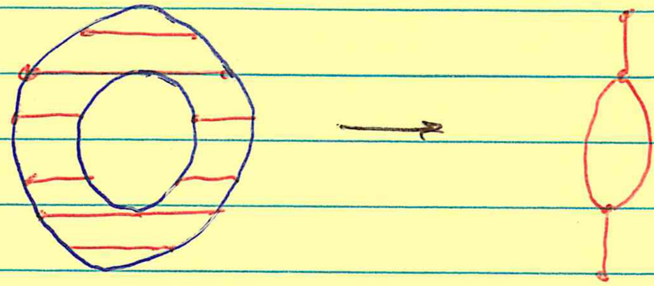
Ex [A quotient of a Hausdorff space need not be H.]

Let $A = \{(x,y) \in \mathbb{R}^2 \mid 1 < x^2 + y^2 < 4\}$; it is an open annulus.
Let $(x_1, y_1) \sim (x_2, y_1)$ if $y_1 = y_2$ and both pairs are on the same open interval of $\mathbb{R} \times \{y_1\} \cap A$.



Using the q-top. on $A/\sim$ the pts a and b cannot be separated by open sets. Likewise for c, d.

(b) If we consider $\bar{A}/\sim$, it is Hausdorff



Det A top. sp. is <u>regular</u> if one pt. sets are closed and $\forall$ pair x, C, $x \notin C$, C closed, $\exists$ disjoint open sets U, V s.t. $x \in U$, $C \subset V$.

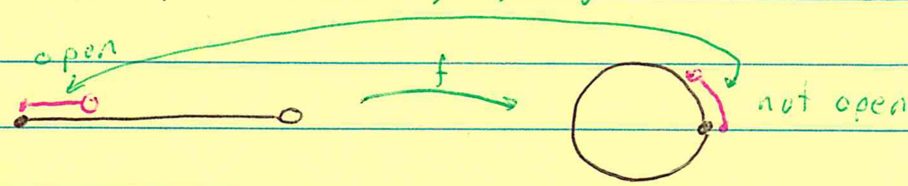
Thm Let X be reg. and $C \subset X$ be closed. Define $x \sim y$ if both are in C. Then $X/\sim$ in the q-top. is Hausdorff.

Pf Let $[x]$ and $[y]$ be members of $X/\sim$, $[x] \neq [y]$. Suppose $x, y \notin C$. Let U' and V' separate x and y in X; $x \in U'$, $y \in V'$, $U' \cap V' = \phi$, both open. Let $U = U' - C$ and $V = V' - C$. Then $x \in U$, $y \in V$, $U \cap V = \emptyset$ and both are open and disjoint from C. Then the images of U and V under the q-map determined by $\sim$ are disjoint open sets in $X/\sim$ containing $[x]$ and $[y]$, resp.

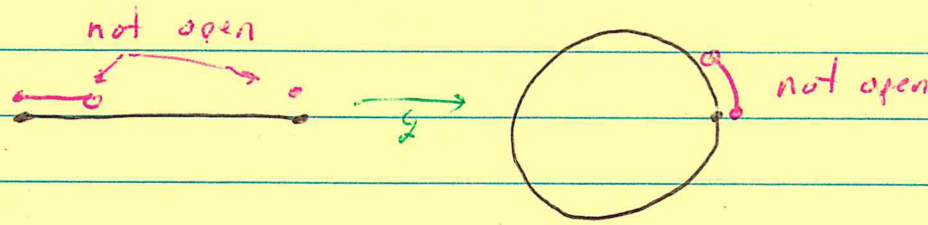
Suppose $x \notin C$ and $y \in C$. $\exists$ U, V, disjoint, open in X with $x \in U$ and $C \subset V$. Then their images under $\sim$ are disj., open with $[x] \in U/\sim$ and $[y] = C \in V/\sim$.

The case $x, y \in C$ need not be considered since then $[x] = C = [y]$. $\square$

Recall Let $f: [0,1) \to S^1$ be given by $f(t) = (\cos(2\pi t), \sin(2\pi t))$. We showed that f is cont., one-to-one, onto, but that f^{-1} is not cont. [Ex 6, §18, pages 106-107.]



But now let $g: [0,1] \to S^1$ be given by the same equation. It is a q-map. To see this let $x \sim y$ in $[0,1]$ iff $x = y$, $x = 0$ and $y = 1$, or, $x = 1$, $y = 0$.



Let $U \subset [0,1]$ be open and saturated. Then either $U \subset (0,1)$ or $\{0,1\} \subset U$. Now $g: (0,1) \to S^1 - \{(1,0)\}$ is a homeo. So, if $U \subset (0,1)$, $f(U)$ is open in $S^1 - \{(1,0)\}$ and hence in S^1. If $\{0,1\} \subset U$, then $\exists\, \varepsilon > 0$ s.t. $[0,\varepsilon) \cup (1-\varepsilon, 1] \subset U$. Since $g([0,\varepsilon) \cup (1-\varepsilon,1])$ and $g(U - \{0,1\})$ are open in S^1, $g(U)$ is open (it is their union). Hence we have a q-map. $\blacksquare$

For $I = [0,1]$ define $x \sim y$ iff $x = y$, or $x = 0, y = 1$ or $x = 1, y = 0$. Let $O = I/\sim = \{\{x\} \mid 0 < x < 1\} \cup \{\{0,1\}\}$. This is exactly the same eq. rel as $x \sim y$ iff $g(x) = g(y)$. Therefore O is homeo to S^1.

Finally, consider $\mathbb{R}/\mathbb{Z}$. Let $x \sim y$ iff $x - y \in \mathbb{Z}$. Then $\mathbb{R}/\mathbb{Z}$ means $\mathbb{R}/\sim$. Each eq. class has a unique member in $[0,1)$. Define $f: \mathbb{R} \to S^1$ by $t \mapsto (\cos(2\pi t), \sin(2\pi t))$, as before. The eq. rel $\sim$ is exactly the same as $x \sim y$ iff $f(x) = f(y)$. Using similar arguments as above, we can show that $\mathbb{R}/\mathbb{Z}$ is homeo to S^1.

Facts Read Thm 22.1 on your own. It addresses what happens when we restrict a q-map to a subspace.

Read Example 7. It shows that $p_i : X_i \to Y_i$, $i = 1, 2$, being q-maps does not imply $p_1 \times p_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is a q-map.

Composites of q-maps are q-maps as you can check.

<u>Thm</u> (22.2) [Important] Let $p: X \to Y$ be a q-map. Let Z be a space with $g: X \to Z$ s.t. g is constant on all sets of the form $p^{-1}(\{y\})$, $y \in Y$. Then $\exists$ a map $f: Y \to Z$ s.t. $f \circ p = g$. Further,

(i) f cont. $\iff$ g cont,

(ii) f q-map $\iff$ g q-map.

<u>Pf</u>

$$\begin{tikzcd} X \arrow[d, "p"'] \arrow[dr, "g"] & \\ Y \arrow[r, dashed, "f"'] & Z \end{tikzcd}$$

Define f as follows. Let $y \in Y$. Consider the eq. class $p^{-1}(\{y\})$ in X. Pick any member, $x \in p^{-1}(\{y\})$. Let $f(y) = g(x)$. By hypothesis it does not matter which $x \in p^{-1}(\{y\})$ we picked, so f is well defined.

(i) Since $g = f \circ p$ and p is cont. if f is cont, then g is cont. Suppose g is cont. Let $V \subset Z$ be open. We will show $f^{-1}(V)$ is open. We know $g^{-1}(V)$ is open in X. But $g^{-1}(V) = p^{-1}(f^{-1}(V))$. Since p is a q-map, $f^{-1}(V)$ is open. Thus, f is cont.

(ii) This part is similar to (i). See textbook. $\boxtimes$

<u>Ex</u>

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "g"] \arrow[d, "p"'] & S^1 \times S^1 \\ \mathbb{R} \times S^1 \arrow[ur, dashed, "f"'] & \end{tikzcd}$$

Let $p(s,t) = (s, (\cos(2\pi t), \sin(2\pi t)))$ and

$g(s,t) = ((\cos(2\pi s), \sin(2\pi s)), (\cos(2\pi t), \sin(2\pi t)))$.

Then $f(s, (x,y)) = ((\cos(2\pi s), \sin(2\pi s)), (x,y))$.

$(x,y) \in S^1$