Chapter 3: Connectedness and Compactness

Take a few minutes to read the chapter introduction. Three of the theorems that form the backbone of calculus are discussed. They are:

1. The Intermediate Value Theorem (IVT)
2. The Maximum Value Thm (or Extreme Value Thm.)
3. The Uniform ~~Convergence~~ Continuity Thm.

Each is about continuous function on a closed bounded interval, $[a, b]$, in $\mathbb{R}$. These thm's can be generalized in useful ways by determining the key properties of $[a, b]$ needed for each proof and "abstracting" them. They are connectedness and compactness.

Ch 3 §23

Connected Spaces

Def Let X be a top. sp. A *separation* of X is a pair of disjoint, nonempty, open subsets whose union is X. In symbols: $\exists\, U, V$ open, $U \cap V = \phi$, $U \cup V = X$. If X has a separation we say that X is *disconnected*. If no separation of X exists, we say X is *connected*.

Ex's Using the subspace topologies $[0,1] \cup [2,3]$, $\{1, 2\}$, $[0,1) \cup (1,2)$ are disconnected. So is $\mathbb{Q}$: Let $U = (-\infty, \pi) \cap \mathbb{Q}$ and $V = (\pi, \infty)$.

$[0,1]$, $[5, \infty)$, and $\mathbb{R}$ are connected but the proof of this is nontrivial. We shall return to these later.

Fact Let Y be a subspace of X. Then Y has a separation iff $\exists\, U, V$ open in X s.t.

$$U \cap Y \neq \phi,\ V \cap Y \neq \phi,\ (U \cap Y) \cap (V \cap Y) = \phi$$
and $Y \subset U \cup V$.

This follows directly from the definitions.

Fact A top. space X is connected iff the only clopen subsets of X and X and $\emptyset$.

Pf Let X be conn. Suppose $U \subset X$ is clopen and $U \neq X$ or $\emptyset$. Let $V = X - U$. Then V is open, nonempty, $U \cap V = \emptyset$ and $U \cap V = X$. But this make U, V a separation of X which contradicts that X is conn.

Let X be a top. space s.t. the only clopen subsets are X and $\emptyset$. Suppose X is disconn. Then $\exists$ a sep. U, V of X. But then U and V are each clopen while neither is X or $\emptyset$. (Check this.) Thus X must be conn. $\boxed{\square}$

Ex's Examples 6 and 7 from the text are...
6. $\mathbb{R}^\omega$ in the box top. is disconn.
7. $\mathbb{R}^\omega$ in the prod. top. is conn.

6. is easy and we do this here. 7 will have to wait until we have a few more tools.

Proof that $\mathbb{R}^\omega$ is disconn. in box top.

Let U = bounded sequences and V = unbounded sequences. Clearly $U \neq \emptyset$, $V \neq \emptyset$, $U \cap V = \emptyset$ and $U \cup V = \mathbb{R}^\omega$. It remains to show they are open.

Let $x = (x_1, x_2, x_3, x_4, \dots) \in U$. We define an open nbhd W of x s.t. $W \subset U$ as follows. Let $W = (x_1 - 1, x_1 + 1) \times (x_2 - 1, x_2 + 1) \times \cdots$. It is open in the box top. and every member of W is bounded. So we have $x \in W \subset U$. Since $x \in U$ was arbitrary we have shown that U is open.

The same argument works for V since if $x = (x_1, x_2, x_3, \dots)$ is unbounded every member of W is unbounded. $\blacksquare$

Some Facts We state some results from the textbook whose proofs are easy, so you should be able to understand them on your own. But ask if you have questions. You may be tested on these.

Lemma 23.2 Suppose $Y \subset X$ is conn. and C, D is a separation of X. Then $Y \subset C$ or $Y \subset D$.

Thm 23.3 Suppose $\{U_\alpha\}$ are all conn. subsets of X and that $\cap U_\alpha \neq \phi$. Then $\cup U_\alpha$ is conn.

Idea of Pf Let $p \in \cap U_\alpha$. Suppose A, B seperate $\cup U_\alpha$. Then $p \in A$ or $p \in B$. Suppose $p \in A$. Show that all $U_\alpha \subset A$. Then $B \cap \cup U_\alpha = \phi$. Same issue if $p \in B$.

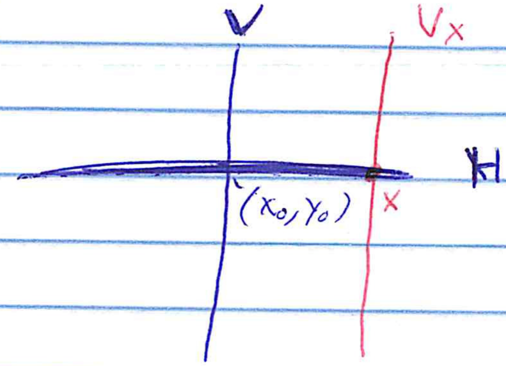
Thm 23.4 Let $A \subset X$ be conn. Then $\bar{A}$ is conn. and if $A \subset B \subset \bar{A}$, then B is conn.

Thm 23.5 Connectedness is a top. property, i.e., if $A \subset X$ is conn. and $f: X \to Y$ is cont., then $f(A)$ is conn.

Idea of pf Let U, V sep. $f(A)$. Then show $f^{-1}(U), f^{-1}(V)$ sep. A.

Thm 23.6 A finite prod. of conn. spaces is conn.

Pf Let X and Y be connected spaces. Let $O = (x_0, y_0) \in X \times Y$. (Think of O as the origin.) Let $H = X \times y_0$, $V = x_0 \times Y$. See picture. H is conn. since it is the image of the cont. map $x \longmapsto (x, y_0)$.



For any $x \in X$ let $V_x = x \times Y$. These are connected. Let $T_x = H \cup V_x$ for each $x \in X$. T_x is conn. since H and V_x are conn. and $(x, y_0) \in H \cap V_x$.

Now consider the union $\bigcup_{x \in X} T_x$. Each T_x contains $O = (x_0, y_0)$. Thus $\bigcup_{x \in X} T_x$ is conn.

But $\bigcup_{x \in X} T_x = X \times Y$. $\boxtimes$

To get finite products use induction.

Back to Exam 7. $\mathbb{R}^\omega$ in the prod. top. is connected.

pf Let $\tilde{\mathbb{R}}^n = \{(x_1, x_2, \dots, x_n, 0, 0, 0 \dots) \mid x_i \in \mathbb{R}\}$.
Then $\tilde{\mathbb{R}}^n$ is the continuous image of $\mathbb{R}^n$.
Just use the mapping $(x_1, x_2, \dots x_n) \mapsto (x_1, x_2, \dots, x_n, 0, 0, \dots)$.
Check this is cont.

Notice that $\mathbb{R}^\infty = \bigcup_{n=1}^{\infty} \tilde{\mathbb{R}}^n$. Since each $\tilde{\mathbb{R}}^n$ contains the origin they have a pt. in common. Hence $\mathbb{R}^\infty$ is conn.

By Exercise #7 in §19 we have $\overline{\mathbb{R}^\infty} = \mathbb{R}^\omega$ using the prod. top. Thus $\mathbb{R}^\omega$ is connected. $\boxed{\square}$