Ch 3 §24

Connected Subspaces of $\mathbb{R}$.

In real analysis courses you likely saw a proof that $\mathbb{R}$, and intervals and rays in $\mathbb{R}$, are connected. Our proof will apply to a broader class of spaces, linear continua.

Recall Def A relation $<$ on a set X is an order relation if

(a) $\forall\, x, y \in X$ either $x=y$, $x<y$, or $y<x$ (exclusively);

(b) if $x<y$ and $y<z$, then $x<z$.

Notation Let X be a set with an order relation. Let $a<b$. Then

$$\left.\begin{array}{l} [a,b] = \{x \mid a \le x \le b\} \\ (a,b] = \{x \mid a < x \le b\} \\ [a,b) = \{x \mid a \le x < b\} \\ (a,b) = \{x \mid a < x < b\} \end{array}\right\} \text{Intervals}$$

$$\left.\begin{array}{l} [a,\infty) = \{x \mid x \ge a\} \\ (a,\infty) = \{x \mid x > a\} \\ (-\infty,b] = \{x \mid x \le b\} \\ (-\infty,b) = \{x \mid x < b\} \end{array}\right\} \text{Rays}$$

Def Let X be a set with an order relation $<$. If X has a least element call it a_0. If X has a greatest element call it b_0. Then $\mathcal{B}$ = all the intervals of the forms $[a_0, b)$, (a,b), $(a,b_0]$, $a_0 < a < b < b_0$, is a basis for the order top. on X.

Def An ordered set L having at least two elements is called a linear continuum if

① L has the least upper bound property (every subset with an upper bd has a l.u.b.)

② If $x<y$, $\exists z \in L$ s.t. $x<z<y$.

Ex's $\mathbb{R}$. Intervals and rays in $\mathbb{R}$.

$\mathbb{Q}$ has ② but not ①.

$\mathbb{Z}$ has ① but not ②.

Thm 24.1 If L is a linear continuum in the order top. then L is connected. (It follows that intervals and rays in L are conn.)

Pf Suppose $L = A \cup B$, each open, disjoint and nonempty. Let $a \in A$ and $b \in B$. Wlog suppose $a<b$. Let $A_0 = A \cap [a,b]$ and $B_0 = B \cap [a,b]$. Then A_0, B_0 is a separation of $[a,b]$.

Let $c = \sup A_0$. We will show that $c \notin A_0$ and $c \notin B_0$! This contradiction then proves the theorem.

Suppose $c \in B_0$. Then $c \neq a$. Either $c = b$ or $a < c < b$. Since B_0 is open in $[a, b]$ (in the subsp. top.) $\exists$ a basic open set (d, d') or $(d, b]$ containing c and in B_0.

$$c \in (d, d') \text{ or } (d, b] \subset B_0.$$

In either case $(d, c) \subset B$. But this means that for every $x \in A_0$ we have $x \leq d$. Thus, d is a upper bound of A_0 that is smaller than c. Thus, $c \notin B$.

Suppose $c \in A_0$. $\exists\, e \in L$ s.t. $[c, e) \subset A_0$. By ② $\exists\, z$ s.t. $c < z < e$. ~~Hence~~ Note $z \in A_0$, and $c < z$. But c was an upper bd of A_0. Thus $c \notin A_0$

But by the lub property, $c \in [a, b]$. This contradiction shows that $[a, b]$ is connected, which implies L is. $\square$

Thm 24.3 (IVT) Let $f: X \to Y$ be continuous, X conn. and Y is an ordered set in the order top. If a and b are two points of X and if $r \in Y$ is s.t.

$$f(a) < r < f(b)$$

then $\exists\, c \in X$ s.t. $f(c) = r$.

Proof Let $A = f(X) \cap (-\infty, r)$ and $B = f(X) \cap (r, \infty)$. They are open subsets of $f(X)$ as a subspace. Since $f(c) \in A$ and $f(b) \in B$, they are nonempty.

Suppose $\nexists\, c \in X$ s.t. $f(c) = r$. Then

$$A \cup B = f(X)$$

and we would have a sep. of $f(X)$. But the continuous image of a connected sp. is connected, so there can be no sep. of $f(X)$. Hence $\exists\, c \in X$ s.t. $f(c) = r$. $\square$

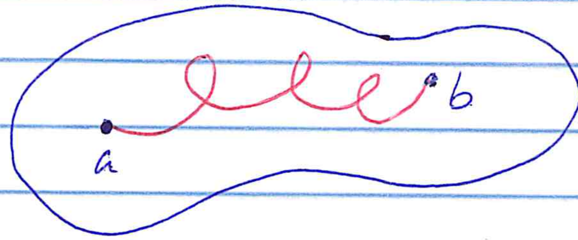
Path Connectedness

Def Let $a, b \in X$. A path in X is a continuous map

$$\gamma : [0,1] \longrightarrow X$$

s.t. $\gamma(0) = a$ and $\gamma(1) = b$.

Note γ need not be 1-to-1 and it is allowed that $a = b$, in which case γ is called a loop.



Def A top. sp. X is path connected if $\forall a, b \in X$ $\exists$ a path γ from a to b.

Thm If X is ~~conn~~ path conn. it is conn.

Pf Suppose X is path conn. but not conn. Let U, V be a separation of X. Let $a \in U$ and $b \in V$. $\exists$ f, a path from a to b.

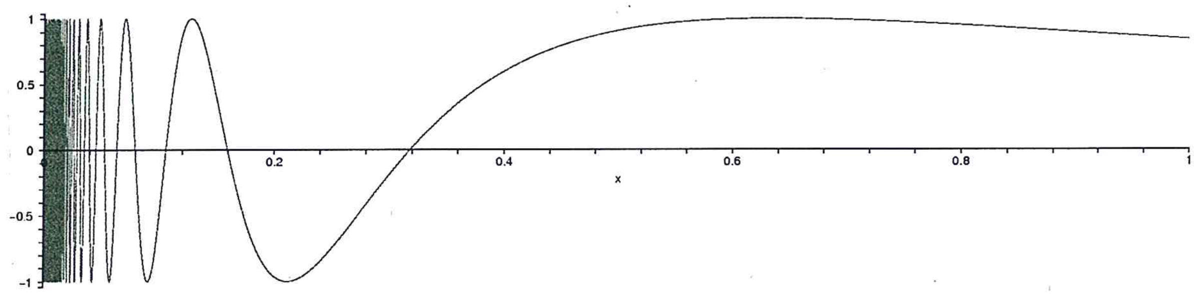
Let $A = f^{-1}(U)$ and $B = f^{-1}(V)$. They are disjoint, open, nonempty and $A \cup B = [0,1]$. Contradiction! $\blacksquare$

The converse is false as the following example shows: conn. $\not\Rightarrow$ path conn.

<u>Ex</u> Recall the Topologist's Sine Curve. Let

$S = \{(x, \sin(\tfrac{1}{x})) \mid 0 < x \leq 1\}$ and $Y = \{0\} \times [-1, 1]$.

We found that $\bar{S} = S \cup Y$. It is clear that S is path conn. Hence S is conn. Thus, $\bar{S}$ is conn. But, we will show that $\bar{S}$ is not path conn.



Suppose $f : [0,1] \to \bar{S}$ is a path from $(0,0)$ to $(1, \sin(1))$. We claim this is impossible.

Since Y is closed (in the subsp. top. of $\bar{S}$) we know that $f^{-1}(Y)$ is closed in $[0,1]$. Thus $f^{-1}(Y)$ is a closed bounded subset of $\mathbb{R}$ and so contains its l.u.b. Call it b. Then $f(b) = (0, y_b)$ for some $y_b \in [-1, 1]$.

We can express $f(t)$ as $(x(t), y(t))$. For $t > b$ we know $y(t) = \sin(1/x(t))$. If (t_n) is a seq. in $[0,1]$ converging to b, then by continuity

$$\lim_{n\to\infty} y(t_n) = y_b.$$

We construct (t_n) as follows. $\exists N \in \mathbb{N}$ s.t. $b + \frac{1}{N} < 1$. ((Clearly $b < 1$.) Define $t_1, t_2, \ldots, t_{N-1}$ any way you like. Let $n \geq N$. $\exists x_n \in (0, x(b + \frac{1}{n}))$ s.t.

$$\sin\left(\frac{1}{x_n}\right) = 1. \qquad (\ast)$$

By the $\boxed{\text{IVT}}$ $\exists t_n \in (b, b + \frac{1}{n})$ s.t. $x(t_n) = x_n$. [To see this notice that $x(b) = 0 < x_n < x(b + \frac{1}{n})$; thus $\exists t_n \in (b, b + \frac{1}{n})$ s.t. $x(t_n) = x_n$.] Therefor,

$$\lim_{n\to\infty} y(t_n) = \lim_{n\to\infty} \sin\left(\frac{1}{x(t_n)}\right) = \lim_{n\to\infty} 1 = 1.$$

Hence $y_b = 1$. But if in $(\ast)$ we had choosen -1 instead of 1, we would have $y_b = -1$. Since y_b cannot take on two different values we have a contradiction. Therefore there is no path is $\bar{S}$ from $(0,0)$ to $(1, \sin(1))$. $\boxtimes$

But we do have the following.

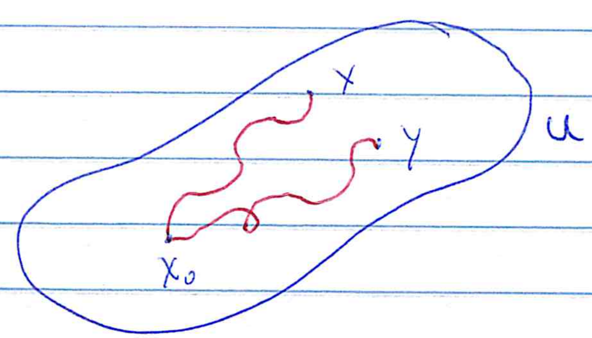
Thm Let $U \subset \mathbb{R}^2$ be open and conn. Then U is path conn.

Pf Assume $U \neq \emptyset$ and let $x_0 \in U$. Let

$$U_0 = \{ x \in U \mid \exists \text{ a path in } U \text{ from } x_0 \text{ to } x \}$$

$$U_1 = U - U_0$$

If $U_1 = \emptyset$ we are done because if there are paths from x_0 to x and from x_0 to y in U we can construct a path from x to y in U. (Use the Pasting Lemma to prove this.)



Suppose $U_1 \neq \emptyset$. We know $U = U_0 \cup U_1$ and $U_0 \cap U_1 = \emptyset$. If both are open we will have a separation of U, a contradiction.

We claim U_0 is open. Let $x \in U_0$. $\exists\, \varepsilon > 0$ s.t. $B(x, \varepsilon) \subset U$ since U is open. Let $y \in B(x, \varepsilon)$. $\exists$ a path from x to y in $B(x, \varepsilon)$ Show this.

Let

$\gamma_1 : [0,1] \to U$ be a path from x_0 to x.

$\gamma_0 : [0,1] \to B(x, \varepsilon)$ be a path from x to y.

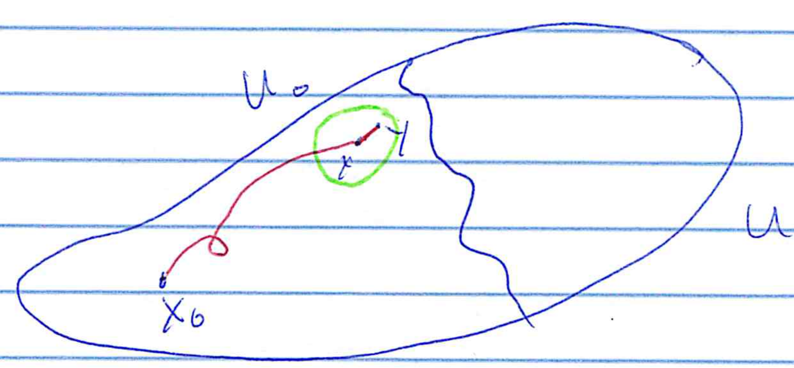
Define

$$\gamma_2 : [0,1] \to U$$

by

$$\gamma_2(t) = \begin{cases} \gamma_1(2t) & t \in [0, \tfrac{1}{2}] \\ \gamma_0(2t-1) & t \in [\tfrac{1}{2}, 1]. \end{cases}$$

Use the Pasting Lemma to show γ_2 is a path from x_0 to y in U. Then by def $y \in U_0$. Thus $B(x, \varepsilon) \subset U_0$. Hence U_0 is open.



But we can show that U_1 is also open. Let $x \in U_1$. $\exists\, \varepsilon > 0$ s.t. $B(x,\varepsilon) \subset U$. Suppose $B(x,\varepsilon) \cap U_0 \neq \phi$. Let $y \in B(x,\varepsilon) \cap U_0$. Then there is a path from x_0 to y and from y to x. As before we can construct a path from x_0 to x in U. Hence $x \in U_0$. That is impossible. Therefor $B(x,\varepsilon) \subset U_1$ and U_1 is open. This contradicts U being connected. $\boxtimes$

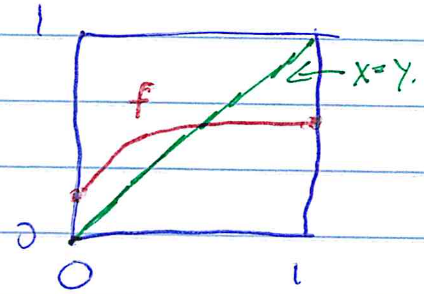
This result is generalized in exercise #4 of Section 25.

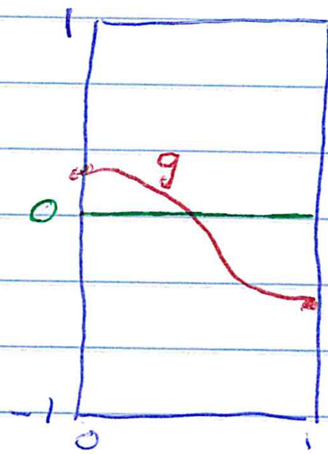
§24 #1c Let $n \geq 2$. Prove that $\mathbb{R}$ and $\mathbb{R}^n$ are not homeo.

Pf Suppose $h : \mathbb{R} \to \mathbb{R}^n$ is a homeo. Then $h : (-\infty, 0) \cup (0,\infty) \to \mathbb{R}^n - \{h(0)\}$ is a homeo by Thm 18.2d. But $(-\infty,0) \cup (0,\infty)$ is not conn. and by Example 4 of this section $\mathbb{R}^n$ can be modified to show $\mathbb{R}^n - \{h(0)\}$ is conn. This contradicts Thm 23.5 applied to h^{-1}.

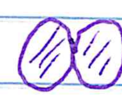
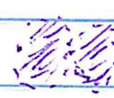
#3 Let $f:[0,1] \to [0,1]$ be cont. Prove that f has a fixed pt., i.e. $\exists x \in [0,1]$ s.t. $f(x) = x$.

Pf Let $a = f(0)$ and $b = f(1)$. Assume $a \neq 0$ and $b \neq 1$. Let $g(x) = f(x) - x$. Then $g:[0,1] \to \mathbb{R}$ is continuous. Now $g(0) = a > 0$ and $g(1) = b-1 < 0$. By the $\boxed{\text{IVT}}$ $\exists c \in (0,1)$ s.t. $g(c) = 0$. Then $f(c) = c$.





#11 (i) If A is conn. in X does it follow that Bd A or Int A are conn.?

No. $Bd([0,1]) = \{0,1\}$. Let $A =$ . Int $A =$ 

(ii) If Int A is conn., does it follow that A is conn.?

No. Consider $A = [1,2] \cup \{3\}$. Int $A = (1,2)$.

(iii) If Bd A is conn., does it follow that A is conn.?

No. Consider $\mathbb{Q} \subset \mathbb{R}$. Bd $\mathbb{Q} = \mathbb{R}$.