§25 Components and Local Connectedness

Def Let X be a top. sp. For $x, y \in X$ define $x \sim y$ if $\exists$ a connected subsp of X that contains x and y. The equivalence classes are called the components of X. Similarly, one can define path components.

Fact $\sim$ is an eq. rel. $x \sim x$ and $x \sim y \Rightarrow y \sim x$ are obvious. To show $x \sim y$ and $y \sim z \Rightarrow x \sim z$, let C_1 contain x and y, and C_2 contain y and z, both conn. Since $C_1 \cap C_2 \neq \phi$, then $C_1 \cap C_2$ is conn. Hence $x \sim z$.

Fact The map from $X \mapsto X/\sim$ given by $x \mapsto$ its component is a q-map.

Thm Components are connected.

Pf Let C be a comp. of X. Let A be a conn. subsp. of X and suppose $A \cap C \neq \phi$. Then we claim $A \subset C$. Assume otherwise. Then A would meet some other comp. C' of X. Now, let $x \in A \cap C$ and $y \in A \cap C'$. But then $x \sim y$ because both are in the conn. sp. A. Now $C = C'$! Thus, if A is conn. it can only meet one component, that is $A \subset C$.

Pick $x_0 \in C$. $\forall x \in C$ we have $x \sim x_0$, so $\exists$ a conn. subsp A_x with $x, x_0 \in A_x$. By the result of the first paragraph $A_x \subset C$.

Notice $C = \bigcup_{x \in C} A_x$ and $x_0 \in \bigcap_{x \in C} A_x$.

Hence C is connected. $\boxed{\cdot}$

<u>Cor</u> We also proved that any conn. subset is within a single component.

<u>Fact</u> Similar results hold for path components.

Local Connectedness

<u>Def</u> A top. sp. X is locally connected at $x \in X$ if for every nbhd U of x, $\exists$ a connected nbhd V of x within U, $x \in V \subset U$. If X is loc. conn. at every pt we say X is locally connected.

There are analogous definitions of locally path conn. at a point and locally path connected.

Note In proving that open connected subsets of $\mathbb{R}^2$ are path connected we used the fact that $\mathbb{R}^2$ is loc. path conn. See exercise #4 in this section.

Ex $[1,2] \cup [3,4]$ is loc. conn. but not conn.

$\{\frac{1}{n} \mid n=1,2,3\ldots\}$ is loc. conn. but $\{\frac{1}{n} \mid n=1,2,3\} \cup \{0\}$ is not.

Recall the topologist's sine curve $\bar{S}$ where $S = \{(x, \sin\frac{1}{x}) \mid x \in (0,1]\}$. S is loc. conn. but $\bar{S}$ is not.

Thm A top. sp. X is loc. (path) conn. iff for every open set $U \subset X$, each (path) comp. of U is open in X.

Pf See Thm 25.3 and 25.4 in textbook.

Thm Let X be a top. sp. Each path comp. of X lies inside a comp of X. If X is loc. path conn., then the comp.'s and path comp's are the same.

Pf See Thm 25.5 in textbook.

#1 What are the components and path components of $\mathbb{R}_\ell$?

We will show that the components of $\mathbb{R}_\ell$ are just the one point sets.

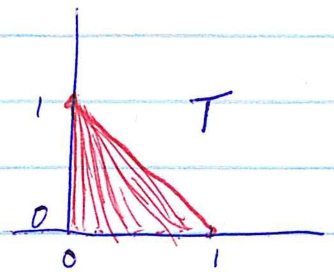
Let $C \subset \mathbb{R}_\ell$ be conn. and suppose it contains two distinct pts a and b. Wlog, suppose $a < b$, Let $c = \frac{a+b}{2}$. Then $(-\infty, c)$, $[c, \infty)$ is a separation of C.

By Thm 25.5 the path comp. of $\mathbb{R}_\ell$ are also the one pt. sets.

Let $f: \mathbb{R} \to \mathbb{R}_\ell$ be cont. What can you say about f?

We will show that f must be a constant function. Since $\mathbb{R}$ is conn. $f(\mathbb{R})$ is connected in $\mathbb{R}_\ell$ and hence is a one pt set.

#5 Let $X = \mathbb{Q} \cap (0,1] \times \{0\} \subset \mathbb{R}^2$ and let T be the union of all line segments from $p = (0,1)$ on the y-axis to points in X on the x-axis. Give T the subsp. top.



ⓐ Show that T is path conn. but is only loc. conn. at p.

Let a and b be two points in T. There is a path from a to p since they are in a common line segment. Likewise there is a path from p to b. Hence we can form a path from a to b in T.

Let U be a nbhd of p. $\exists$ $\epsilon > 0$ s.t. $B(p, \epsilon) \subset U$. $T \cap B(p, \epsilon)$ is path conn. by the same argument used above.

Let $x \in T - \{p\}$. Let $0 < \epsilon < \frac{d(p,x)}{2}$. Then

$B(x, \epsilon)$ has infinitely many components.

(6) Find a subsp. of $\mathbb{R}^2$ that is path conn. but is not loc. conn at any pt.