§26 Compact Spaces

Def A collection $\mathcal{A}$ of subsets of a top. sp. X is said to cover X or to be a covering of X, if $\bigcup \mathcal{A} = X$. We will be mostly interested in open coverings in which each set is open.

Def A top. sp. X is compact if every open covering of X contains a finite subcollection that also covers X. (A subcoll. of a cover that is still a cover of X is called a subcover.)

Ex Not compact: $(0,1)$, $\mathbb{R}$, $\mathbb{R}^2$, $\{\frac{1}{n} \mid n=1,2,3\ldots\}$.

Compact: $[0,1]$, S^1, T^2, S^2, $\mathbb{R} \cup \{\infty\}$ (one pt compactification), $\{0\} \cup \{\frac{1}{n} \mid n=1,2,3,\ldots\}$.

Fact A subsp. Y of X is compact iff every open cover of Y by subsets of X has a finite subcover.

Pf See Lemma 26.1 in text.

Thm (26.2) Every closed subsp. of a compact sp. is compact.

Pf Let X be compact and $Y \subset X$ be closed. Let $\mathcal{A}$ be an open cover of Y. Let $\mathcal{B} = \mathcal{A} \cup \{X - Y\}$; it is an open cover of X. Let $\mathcal{B}'$ be a finite subcover of $\mathcal{B}$. Let $\mathcal{A}' = \mathcal{B}' - \{X-Y\}$. It is a finite subcover from $\mathcal{A}$. $\square$

Thm (26.3) Every compact subsp of a Hausdroff sp. is closed.

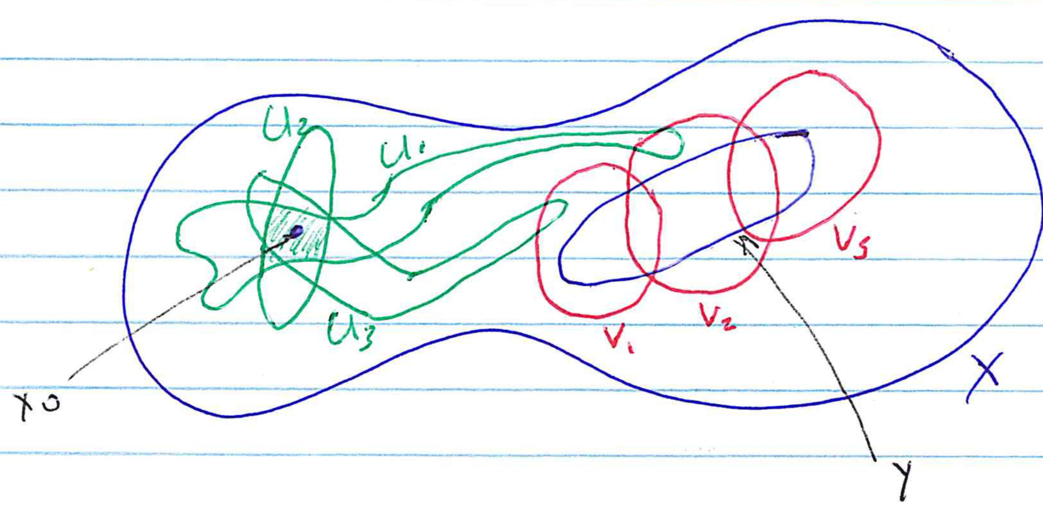
Pf Let Y be a compact subsp. of a Hausdroff sp. X.
Let $x_0 \in X - Y$. (If $X - Y = \phi$, we are done.)
$\forall y \in Y$ choose disjoint nbhds U_y of x and V_y of y.
Then the collection $\mathcal{V} = \{V_y \mid y \in Y\}$ is an open cover of Y. There exists a finite subcover

$$\mathcal{V}' = \{V_{y_1}, V_{y_2}, \ldots, V_{y_n}\}.$$

Let $U = U_{y_1} \cap \cdots \cap U_{y_n}$ and $V = V_{y_1} \cup \cdots \cup V_{y_n}$. Then $x_0 \in U$, $Y \subset V$, both U and V are open and $U \cap V = \phi$. (Why?). Thus $U \cap Y = \phi$ and we have

$$x_0 \in U \subset X - Y.$$

Hence $X - Y$ is open and Y is closed. $\square$



Thm (26.5) Compactness is a top property, i.e. the continuous image of a compact space is compact.

Outline of Pf: Start with an open cover of $f(C)$. Pull it back to get an open cover of C. Pass to a finite subcover. The corresponding finite subcollection of the original cover of $f(C)$ will be a cover of $f(C)$. See textbook for details. ∎

Def A collection $\mathcal{C}$ of subsets of a top. sp. X has the finite intersection property or FIP if for every finite subcollection $\{C_1, C_2, \ldots, C_n\} \subset \mathcal{C}$ we have $C_1 \cap C_2 \cap \cdots \cap C_n \neq \emptyset$.

Ex $\{(0, \frac{1}{n})\}_{n=1}^{\infty}$, $\{[0, \frac{1}{n}]\}_{n=1}^{\infty}$, $\{[n, \infty)\}_{n=1}^{\infty}$ are FIP.

$\{[n, n+1]\}_{n=0}^{\infty}$ is not FIP.

Thm (26.9) Let X be a top. sp. Then X is compact iff for every collection $\mathcal{C}$ of closed sets having the FIP we have $\bigcap \mathcal{C} \neq \emptyset$.

Ex

$\bigcap_{n=1}^{\infty} (0, \frac{1}{n}) = \emptyset$,	$\bigcap_{n=1}^{\infty} [0, \frac{1}{n}] = \{0\}$,	$\bigcap_{n=1}^{\infty} [n, \infty) = \emptyset$.
$X = (0, 1)$	$X = [0, 1]$	$X = (0, \infty)$.
not closed		not bounded.

Pf Let X be a top. sp. Notice, for every collection $\mathcal{A}$ of open sets, there is a dual collection of closed sets given by $\mathcal{C} = \{X - A \mid A \in \mathcal{A}\}$. Note also that $\mathcal{A}$ covers X iff $\mathcal{C}$ has empty intersection since $X - \bigcup_{A\in\mathcal{A}} A = \bigcap_{A\in\mathcal{A}} (X-A) = \bigcap_{C\in\mathcal{C}} C$.

X compact $\Leftrightarrow$ Given a collection $\mathcal{A}$ of open subsets of X, if $\mathcal{A}$ covers X, then $\exists$ a finite subcollection that covers X.

$\Leftrightarrow$ Given a collection $\mathcal{A}$ of open subsets of X, if no finite subcollection covers X, then $\mathcal{A}$ does not cover X.

$\Leftrightarrow$ Given a collection $\mathcal{C}$ of closed subsets of X, if every finite intersection is nonempty, then the intersection of all members of $\mathcal{C}$ is nonempty.

$\Leftrightarrow$ FIP $\Rightarrow \bigcap \mathcal{C} \neq \emptyset$. $\square$

Corollary: Let $C_1 \supset C_2 \supset C_3 \supset \ldots$ be a nested sequence of nonempty, closed, compact sets. Then their intersection is nonempty.

Thm (26.6) Let $f: X \to Y$ be a cont. bijection. If X is compact and Y is Hausdorff, then f^{-1} is cont. and so f is a homeo.

Pf To prove f^{-1} is cont. we show that $f = (f^{-1})^{-1}$ takes closed ~~open~~ sets to closed ~~open~~ sets. Let $C \subset X$ be closed. Since X is compact, C is compact. Thus $f(C) \subset Y$ is compact. Since Y is Hausdroff $f(C)$ is closed. ∎

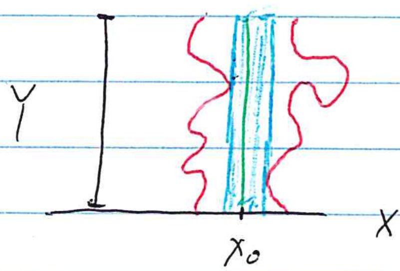
Recall $f: [0,1) \to S^1$ given by $t \mapsto (\cos(2\pi t), \sin(2\pi t))$ is one-to-one, onto and cont., but is not a homeo. Note that $[0,1)$ is not compact.

Ex Prove that $x^{1/3}: \mathbb{R} \to \mathbb{R}$ is cont.

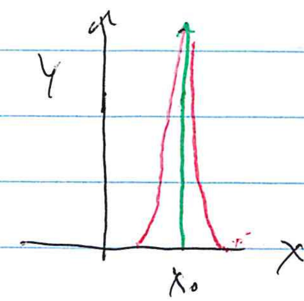
Pf By def. $x^{\frac{1}{3}}$ is the inverse of $x^3: \mathbb{R} \to \mathbb{R}$, which is cont., one-to-one and onto. But $\mathbb{R}$ is not compact. Let $y_0 \in \mathbb{R}$ and $x_0 = y_0^{1/3}$. $\exists\, n \in \mathbb{N}$ s.t. $x_0 \in [-n, n]$. Now x^3 is cont., 1-1, onto as a map from $[-n, n] \to [-n^3, n^3]$. $y_0 \in [-n^3, n^3]$. By Thm 26.6 $y^{1/3}: [-n^3, n^3] \to [-n, n]$ is cont. Thus $y^{1/3}$ is cont. at y_0. Since y_0 was arbitrary $y^{1/3}$ is cont on $\mathbb{R}$. ∎

We will prove that the cross product of two compact spaces is compact. A key step is

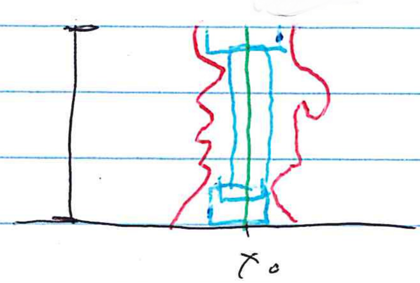
Lemma (26.8) **The Tube Lemma** Consider the product space $X \times Y$, where Y is compact. If $N \subset X \times Y$ is open and $\{x_0\} \times Y \subset N$ for some $x_0 \in X$, then N contains an "open tube," that is $\exists$ an open nbhd $W \subset X$ of x_0 s.t. $W \times Y \subset N$.



(False if Y is not compact.)



Pf Let $Y_0 = \{x_0\} \times Y$; it is compact since it is homeo to Y. Construct an open cover of Y_0 using basis elements that lie in N. (We know N is a union of basis elements, so we can do this.) Pass to a finite subcover and discard any sets that miss Y_0. Enumerate these $U_1 \times V_1, U_2 \times V_2, \dots, U_n \times V_n$. Each U_i is open in X, and $x_0 \in U_i$. Each V_i is open in Y. Let $W = \bigcap_{i=1}^{n} U_i$. Then $x_0 \in W \subset X$ and W is open. Thus $W \times Y$ is open in $X \times Y$, ~~covers~~ contains Y_0 and $W \subset N$. $\blacksquare$



Thm (26.7) The product of two compact spaces is compact.

Cor By induction, finite prod's of compact sp's are compact.

Pf Let X and Y be compact. Let $\mathcal{A}$ be an open cover of $X \times Y$. Let $x \in X$ and $Y_x = \{x\} \times Y$. $\exists$ a finite subcollection from $\mathcal{A}$ the covers Y_x. Call them $A_1^x, A_2^x, \ldots, A_{m_x}^x$. Let $N_x = A_1^x \cup \cdots \cup A_{m_x}^x$. By the $\boxed{\text{Tube Lemma}}$ $\exists\, W_x \subset X$, a nbhd of x_0 s.t.

$$Y_x \subset W_x \times Y \subset N_x.$$

Do this for each $x \in X$. Then $\{W_x \mid x \in X\}$ is an open cover of X. Let $\{W_{x_1}, W_{x_2}, \ldots W_{x_n}\}$ be a finite sub cover of X. For $i = 1, \ldots n$ we have

$$\{A_1^{x_i}, A_2^{x_i}, \ldots, A_{m_{x_i}}^{x_i}\} \subset \mathcal{A}$$

covers $W_{x_i} \times Y$ since it is a cover of N_{x_i}.

Therefore $\{A_j^{x_i}\}_{i=1, j=1}^{n \quad m_{x_i}} \subset \mathcal{A}$ is a finite cover of $X \times Y$. ☐

In Ch. 5, Section 37, it will be proven that arbitrary products of compact spaces are compact; This result is known as the Tychonoff Theorem.