Section 27

Thm (27.1) $[0,1]$ is compact.

Pf Let $\mathcal{A}$ be an open cover of $[0,1]$. (open in subsp top)

Claim: If $x \in [0,1)$, $\exists\, y \in (x,1]$ s.t. $[x,y]$ is covered by one element of $\mathcal{A}$.

Pf: Let $A \in \mathcal{A}$ with $x \in A$. A is open so $\exists\, \varepsilon > 0$ s.t. $(x-\varepsilon, x+\varepsilon) \subset A$ (this uses that $x \neq 1$). Let $y = x + \varepsilon/2$. Then $[x,y] \subset A$.
(or $[0,\varepsilon)^c$ if $x=0$)

Let C be the set of all pts $z \in (0,1]$ s.t. $[0,z]$ can be covered by finitely many members of $\mathcal{A}$. By the claim with $x=$ C is not empty. Let $c = \text{lub}\, C$. Clearly $c \in (0,1]$. We will show $c = 1$.

First we show $c \in C$. Suppose not. Let $A \in \mathcal{A}$ be s.t. $c \in A$. $\exists\, d \in (0,c)$ s.t. $(d,c] \subset A$. Since $c \notin C$, $\exists\, z \in C$ in (d,c); otherwise d would be a smaller upper bound. Thus $[0,z]$ can be covered by finitely many members of $\mathcal{A}$ (by def. of C) and $[z,c]$ can be covered by one member of $\mathcal{A}$. Thus $[0,c]$ can be covered by finitely many members of $\mathcal{A}$. Hence $c \in C$. This contradicts $c \notin C$! Thus $c \in C$.

Finally we show $c=1$. Suppose $c<1$. By the claim $\exists\, y \in (c,1]$ s.t. $[c,y)$ can be covered by a single member of $\mathcal{A}$. Thus $[0,y]$ can be covered by finitely many members of $\mathcal{A}$. Thus $y \in C$, contradicting that $c = \operatorname{lub} C$. Thus $c=1$.

$\blacksquare$

~~Cor. Every closed bounded interval in $\mathbb{R}$ is compact.~~
~~closed Boxes in $\mathbb{R}^n$ are compact. What about closed balls in $\mathbb{R}^n$? $\subset$ Box.~~

~~Recall: For $x=(x_1,x_2,\ldots,x_n)$, $y=(y_1,y_2,\ldots,y_n)$ in $\mathbb{R}^n$~~

~~$d(x,y) = \sqrt{\sum_{i=1}^{n}(x_i-y_i)^2}$ $\quad \rho(x,y) = \max\{|x_i - y_i|\}$.~~

~~Notice that $\rho(x,y) \le d(x,y) \le \sqrt{n}\,\rho(x,y)$.~~
~~So a set $A \subset \mathbb{R}^n$ is bdd in d iff it is bdd in ρ, and the topologies are the same.~~

<u>Corallary</u> Every closed bounded interval in $\mathbb{R}$ is compact.
All closed boxes in $\mathbb{R}^n$ are compact.
All closed balls in $\mathbb{R}^n$ are compact.

Thm 27.3 A subspace $A \subset \mathbb{R}^n$ is compact iff it is closed and bdd

Pf Let $A \subset \mathbb{R}^n$ be compact. Since $\mathbb{R}^n$ is Hausdorff A is closed.
Let $\mathcal{B} = \{ B_d(0, m) \mid m \in \mathbb{Z}_+ \}$. Then $\mathcal{B}$ is an open cover
open ball

of $\mathbb{R}^n$ and hence of A. Since A is compact $\exists$ a finite subcover $\mathcal{B}'$. Let $M \in \mathbb{Z}_+$ be s.t. $B_d(0, M)$ is the largest member of $\mathcal{B}'$. It contains all the others so $A \subset B_d(0, M)$.
Hence A is bdd.

Suppose A is closed and bdd. $\exists$ $N \in \mathbb{Z}_+$ s.t.

$$A \subset [-N, N]^n$$

Since boxes are compact and A is closed, A is compact.

∎

Thm 27.4 Extreme Value Thm

Let $f: X \to Y$ be cont., where X is compact and Y is an ordered set in the order top.
Then $\exists\, c, d \in X$ s.t.

$$f(c) \leq f(x) \leq f(d) \quad \forall x \in X.$$

Note: It can happen that $c = d$, and neither needs to be unique.

Pf Let $A = f(X) \subset Y$. If $f(X)$ is finite it is compact.
Suppose A has no ~~upper bound~~ max. element.
Let $\mathcal{A} = \{(-\infty, a) \mid a \in A\}$.

It is an open cover of A: Let $x \in A$. $\exists\, a \in A$, $a > x$.
Hence $x \in (-\infty, a) \in \mathcal{A}$.
Suppose $\{(-\infty, a_1), \dots (-\infty, a_n)\}$ is any finite subcollection and suppose $a_1 \leq a_2 < \dots < a_n$. Then a_n is not covered. This contradicts A being compact. Hence A must have a largest (max) element.

The existence of a min is similar.

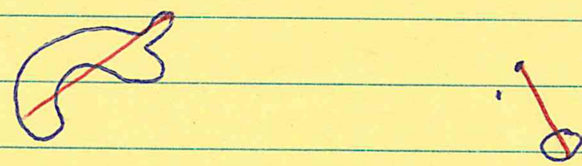
$\square$

Def Let (X,d) be a metric space.

Let $A \subset X$. $A \neq \emptyset$ Then the <u>diameter</u> of A is

$$\operatorname{diam}(A) = \sup\{ d(x,y) \mid x,y \in A \}.$$

Ex



$$\operatorname{diam}(\mathbb{R}) = \infty.$$

Def Let (X,d) be a metric space, $A \subset X$, $A \neq \emptyset$ and $x \in$
Then the distance from ~~x~~ to A is

$$d(x,A) = \inf\{ d(x,a) \mid a \in A \}.$$

Ex $X = \mathbb{R}$. $d(1,(0,1)) = 0$, $d(1,(4,8]) = 3$

Thm For a fixed set $A \subset X$, $A \neq \emptyset$, we can regard $d(x,A)$ as a function from X to $\mathbb{R}$. It is cont.

Pf Let $x_0 \in X$ and $\varepsilon > 0$. Let $\delta = \varepsilon$. Suppose $x \in X$ with $d(x_0,x) < \delta$. We will show that

$$|d(x_0,A) - d(x,A)| \leq d(x_0,x) < \delta = \varepsilon.$$

$\forall a \in A$, $d(x_0, A) \leq d(x_0, a) \leq d(x_0, x) + d(x, a)$.

Thus, $d(x_0, A) - d(x_0, x) \leq \inf\limits_{a \in A} d(x, a) = d(x, A)$

Hence, $d(x_0, A) - d(x, A) \leq d(x_0, x)$.

We can reverse the roles of x_0 and x and get

$$d(x, A) - d(x_0, A) \leq d(x, x_0).$$

Thus

$$\left| \qquad \right| < \varepsilon.$$

∎

<u>Thm 27.5</u> (Lebesgue number lemma)
Let $\mathcal{A}$ be an open covering of a metric sp (X, d). If X is compact, $\exists \delta > 0$ s.t. $\forall B \subset X$ with $\operatorname{diam}(B) < \delta$, $\exists A \in \mathcal{A}$ s.t. $B \subset A$. ~~The largest~~ Such a δ is called ~~the~~ a Lebesgue number ~~of~~ for $\mathcal{A}$.

<u>Ex</u> $X = [0, 1]$. $\mathcal{A} = \{[0, \tfrac{2}{3}), (\tfrac{1}{3}, 1]\}$
Then $\delta = \tfrac{1}{2}$ will work.

$X = \mathbb{R}$. Let ~~[illegible]~~.

Let $I_0 = (-1, 1)$, $I_n = (n - |\tfrac{1}{n}|, n + 1 + |\tfrac{1}{n}|)$ $|n| > 0$.
Let $\mathcal{A} = \{I_n \mid n \in \mathbb{Z}\}$. Then no $\delta > 0$ will work.

Pf Assume $X \notin \mathcal{A}$, for otherwise the result is trivial.

Let $\{A_1, \dots, A_n\} \subset \mathcal{A}$ be a finite subcovering. Let $C_i = X - A_i$, $i = 1 \dots n$ and then define $f: X \to \mathbb{R}$ by

$$f(x) = \frac{1}{n} \sum_{i=1}^{n} d(x, C_i) \quad \leftarrow \text{an average}$$

It is easy to see that $f(x) > 0 \quad \forall x \in X$: $\forall i$ s.t. $x \in A_i$, $\exists \epsilon > 0$ s.t. $B(x, \epsilon) \subset A_i$. Thus $d(x, C_i) \geq \epsilon$.

Since X is compact and $\mathbb{R}$ is ordered, $\exists$ a minimum value of f. Call it δ. $\delta > 0$ since it is a minimum value of f (not just an inf).

Let $B \subset X$ have $\operatorname{diam}(B) < \delta$. Let $x_0 \in B$. Then

$$\delta \leq f(x_0) \leq d(x_0, C_m),$$

where $d(x_0, C_m)$ is the largest member of $\{d(x_0, C_i)\}_{i=1}^{n}$

Then $\quad B \subset X - C_m \subset A_m$.

$\boxed{\text{///}}$

Hwk, Extra Credit: Let $X = [0, 1]$. Let $\mathcal{A} = \{[0, \frac{1}{3}), (\frac{1}{4}, \frac{3}{4}), (\frac{2}{3}, 1]\}$. Compute and plot the graph of f over X. What is δ?

Def A function $f : X \to Y$, X, Y metric spaces, is <u>uniformly continuous</u> if $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. $\forall x_0, x_1 \in X$

$$d_X(x_0, x_1) < \delta \implies d_Y(f(x_0), f(x_1)) < \varepsilon.$$

The point is δ depends only on ε and not on x_0, x_1. The amount of stretching is "uniform."

Thm Let $f : X \to Y$ be cont with X and Y metric spaces. If X is compact then f is uniformly cont.

Pf. Let $\varepsilon > 0$. Cover Y with open balls of radius $\varepsilon/2$, $\{B(y, \varepsilon/2) \mid y \in Y\}$. Let

$$\mathcal{U} = \{f^{-1}(B(y, \tfrac{\varepsilon}{2})) \mid y \in Y\}.$$

This is an open cover of X. Since X is compact it has a L. num. $\delta > 0$. If $x_0, x_1 \in X$ and $d(x_0, x_1) < \delta$ then the set $\{x_0, x_1\}$ has diam $< \delta$. By L.N.L $\{x_0, x_1\} \subset f^{-1}(B(y, \varepsilon/2))$ for some y. Thus

$$f(x_0), f(x_1) \text{ are both in } B(y, \varepsilon/2).$$

Hence $d(f(x_0), f(x_1)) < \varepsilon$. $\blacksquare$

Def If X is a top. sp. a point $p \in X$ is isolated if $\{p\}$ is open in X.

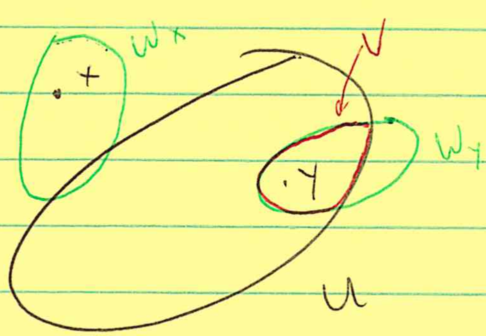
Ex $X = [0,1] \cup \{2\}$. Then 2 is the only isolated pt.

Thm 27.7 Let X be a nonempty, compact, Hausdorff space. If X has no isolated points, then X is uncountable.

Note Before starting the proof, you should see if you can show that X is not finite.

Pf Step 1 Let $x \in X$ and $U \subset X$ be open and nonempty. (Maybe x is in U, maybe not.) We claim $\exists$ an open set V s.t. $V \subset U$ and $x \notin \overline{V}$.

To see this choose $y \in U$, $y \neq x$. (If $x \notin U$, such a y exists since $U \neq \emptyset$. If $U = \{x\}$, then x is isolated. Hence such a y exists.) Since X is Hausdorff $\exists$ open disjoint nbhds of x and y, W_x and W_y respectively. Let $V = U \cap W_y$. Since $y \in V$, $V \neq \emptyset$. Also x is not a limit point of V since $W_x \cap V = \emptyset$.



Step 2 Given any function $f: \mathbb{N} \to X$ we will show it is not onto. (Hence X is not countable.)
Let $x_n = f(n)$.

Let $U_1 = X$. Apply Step 1 to x_1, U_1 to produce V_1.
Let $U_2 = V_1$. Apply Step 1 to x_2, U_2 to produce V_2.
Let $U_3 = V_2$. Apply Step 1 to x_3, U_3 to produce V_3.
Etc.

Thus, we have $X \supset V_1 \supset V_2 \supset V_3 \supset V_4 \supset \cdots \supset V_n \supset \cdots$
so

$$X \supset \overline{V}_1 \supset \overline{V}_2 \supset \overline{V}_3 \supset \overline{V}_4 \supset \cdots \supset \overline{V}_n \supset \cdots$$

Since X is compact $\bigcap_{i=1}^{\infty} \overline{V}_i \neq \emptyset$. Let $y \in \bigcap_{i=1}^{\infty} \overline{V}_i$.

We have $x_1 \notin \overline{V}_1$. So $y \neq x_1$.

We have $x_2 \notin \overline{U}_2$ so $y \neq x_2$

And so on. Since $x_n \notin \overline{U}_n$ for all $n = 1, 2, 3, \ldots$

$y \neq x_n$ for any $n = 1, 2, 3 \ldots$. Thus, f is not onto. $\blacksquare$