§28 Limit Point Compactness, Sequential Compactness, ...

Def A top. sp. X is *limit point compact* if every infinite subset of X has a limit pt.

Def A top. sp. X is *sequentially compact* if every seq. in X has a convergent subseq.

Thm Compact $\Rightarrow$ L.P.C. (Examples 1,2 and 3 in the textbook show L.P.C. $\not\Rightarrow$ Compact.)

Pf Let X be compact. Let $A \subset X$. Suppose A has no limit pts. We will show that A is finite. It is clear that A is closed.

$\forall\, a \in A$, $\exists$ nbhd U_a s.t. $U_a \cap A = \{a\}$. Then $\mathcal{U} = \{U_a \mid a \in A\} \cup \{X-A\}$ is an open cover of X. Let $\mathcal{U}'$ be a finite subcover of X. $\exists\, a_1, a_2, \ldots a_n$ in A s.t. $\{U_{a_1}, U_{a_2}, \ldots, U_{a_n}\} \cup \{X-A\}$ covers X. Then $A \subset \bigcup_{i=1}^{n} U_{a_i}$.

But $A \cap \bigcup_{i=1}^{n} U_{a_i} = \{a_1, a_2, \ldots, a_n\}$.

Therefore, $A = \{a_1, a_2, \ldots, a_n\}$ and we see that A is finite. $\blacksquare$

Thm Let X be a S.C. metric space. Then the Leb. num. lem. holds, that is, if $\mathcal{A}$ is an open cover of X, $\exists \delta > 0$ s.t. $\forall$ subset C of X with diam $< \delta$, $\exists A \in \mathcal{A}$ s.t. $C \subset A$.

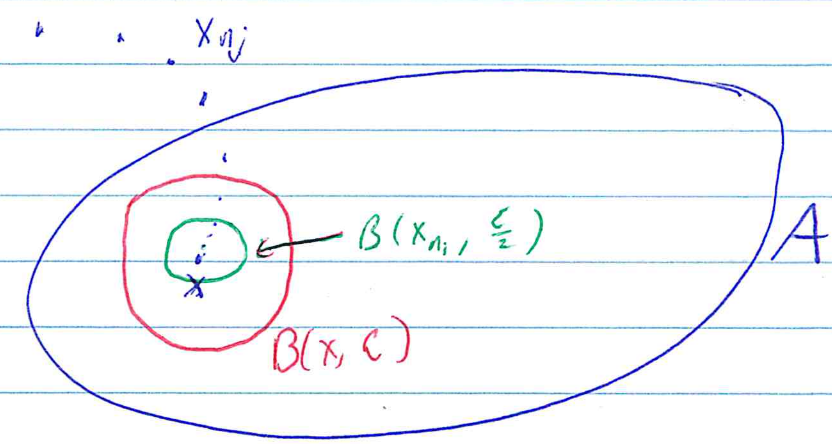
Pf Suppose this is false. Then $\forall n > 0$ $\exists C_n \subset X$ with diam $< \frac{1}{n}$ s.t. C_n is not a subset of any member of $\mathcal{A}$.

Let $x_n \in C_n$, $n = 1, 2, 3 \ldots$. $\exists$ a convergent subseq, $x_{n_i} \to x$.
$\exists A \in \mathcal{A}$ s.t. $x \in A$ since $\mathcal{A}$ is a cover.
$\exists \varepsilon > 0$ s.t. $B(x, \varepsilon) \subset A$ since A is open.

$\exists N$ s.t. $n_i > N \Rightarrow x_{n_i} \in B(x, \frac{\varepsilon}{2})$.
If $n_i > 2/\varepsilon$, then $C_{n_i} \subset B(x_{n_i}, \varepsilon/2)$.
Thus, for $n_i > \max\{N, 2/\varepsilon\}$ we have

$$C_{n_i} \subset B(x_{n_i}, \tfrac{\varepsilon}{2}) \subset B(x, \varepsilon) \subset A.$$

This contradiction proves the thm. $\square$



Note We did this in 452. See Ch 2, Sec 7 of Real Math. Analysis by Pugh. It is also in the proof of Thm 28.2 in Munkres.

Thm Let X be a S.C. metric sp. Then $\forall \varepsilon > 0$, $\exists$ a finite cover of X by open balls of radius ε.*

pf Suppose instead $\exists\, \varepsilon > 0$ s.t. X cannot be covered by finitely many ε-balls. Let $x_1 \in X$. Since $B(x_1, \varepsilon)$ does not cover X we can pick $x_2 \in X - B(x_1, \varepsilon)$. Since $\{B(x_1, \varepsilon), B(x_2, \varepsilon)\}$ does not cover X we can pick $x_3 \in X - B(x_1, \varepsilon) - B(x_2, \varepsilon)$. And so on. This generates a seq (x_n).

But (x_n) has no convergent subseq. for suppose $x_{n_i} \to x \in X$. Then $B(x, \frac{\varepsilon}{2})$ contains at most one member of the seq. $\square$

*This is called <u>total boundedness</u> in Pugh's book, pg 103.

Thm Let X be a metric sp. Then $C \Leftrightarrow$ L.P.C. $\Leftrightarrow$ S.C.

Pf We know $C \Rightarrow$ L.P.C. Now we show L.P.C. $\Rightarrow$ S.C. when X is a metric sp. Let (x_n) be a seq. in X. Let S = the underlying set of (x_n). If S is finite then (x_n) has a constant subseq which obviously converges. Suppose S is infinite. It has a L.P. x. We will find a subseq (x_{n_i}) that converges to x.

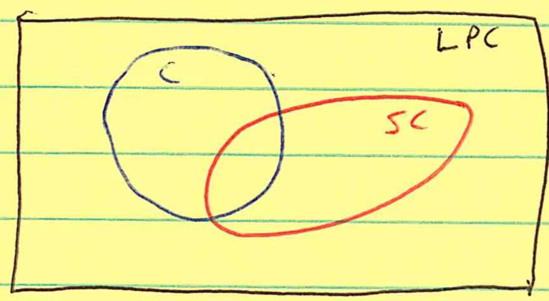
$\forall k > 0$ the open ball $B(x, \frac{1}{k})$ contains infinitely many members of S. (Recall Metric $\Rightarrow$ Hausdorff, then use Thm 17.9 on page 99.)

Let $k=1$. Pick some $x_{n_1} \in B(x, 1)$. For $k=2$, pick $n_2 > n_1$ with $x_{n_2} \in B(x, \frac{1}{2})$. Continue in this way. Then the subseq (x_{n_i}) converges to x.

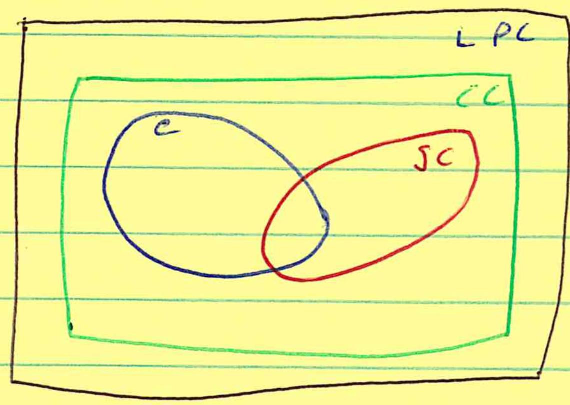
Finally we show S.C. $\Rightarrow C$ when X is a metric sp. Let $\mathcal{A}$ be an open cover of X and let $\delta > 0$ be a Leb. num. for $\mathcal{A}$. $\exists$ a finite cover of X by open balls of radius $\delta/3$, $\{B_1, B_2, \dots, B_n\}$. Since diam $B_i = \frac{2}{3}\delta < \delta$ $\exists$ $A_i \in \mathcal{A}$ with $B_i \subset A_i$ for $i = 1, \dots, n$. Then $\{A_1, \dots, A_n\} \subset \mathcal{A}$ covers X. $\square$

Facts For general top. spaces....

$C \Rightarrow LPC$	[Thm 28.1]
$LPC \not\Rightarrow C$	[Examples 1, 2, and 3]
$C \not\Rightarrow S.C.$	[GT 17G.1, CET 105.5]
$SC \not\Rightarrow C$	[GT 17G.2, CET 43.8]
$LPC \not\Rightarrow S.C.$	[CET 106.2]
$SC \Rightarrow LPC$	[Proof is extra credit!]



Exercise #4 introduces Countable compactness (CC).



GT = General Topology by Willard.
CET = Counterexamples in Topology by Steen & Seebach.