§29 Local Compactness and 1 pt. Compactifications

Def A top. sp. X is locally cpt. at $x \in X$ if x has a compact nbhd ($x \in U \subset C$, U open, C cpt.). The sp. X is loc. cpt. if it is loc. cpt. at each $x \in X$.

Note This is different than how we defined loc. connectedness. But we do have the following.

Thm (29.2) Let X be a Hausdorff space. Then X is loc. cpt iff $\forall x \in X$ and nbhd U of x, $\exists$ nbhd V of x s.t. $\bar{V} \subset U$, and $\bar{V}$ is compact.

We delay the proof until after we develop the theory of one-pt-compactifications.

Def If $A \subset X$ and $\bar{A} = X$, the A is dense in X.

Def If Y is a compact Hausdorff sp and $X \subsetneq Y$ with $\bar{X} = Y$, the Y is called a compactification of X. If $Y - X$ is a single pt, then Y is called the one-pt-compactification of X.

Ex $\mathbb{R}^n \cup \{\infty\}$ with basis the open balls in $\mathbb{R}^n$ and sets of the form $\mathbb{R}^n \cup \{\infty\} - K$, where K is a compact subset of $\mathbb{R}^n$. Check that this is a basis.

Q: When do one-pt-comp...'s exist?

Thm (29.1) Let X be a top. sp. that is not cpt.
ⓐ If X is loc. cpt. and H., then it has a 1-pt-comp... .
ⓑ If X has a 1-pt-comp..., then it is loc. cpt. and H.
ⓒ 1-pt-compactifications are unique: If Y and Y' are 1-pt-com... of X, then they are homeomorphic and $\exists$ a homeo that is the identity on X.

Pf ⓐ Let $Y = X \cup \{\infty\}$ and $\mathcal{T}_Y = \{U \mid U \subset X \text{ is open}\} \cup \{Y - K \mid K \subset X \text{ is compact}\}$. It is easy to check that this gives a topology; see textbook.

We need to check that X, with its given top., is a subsp. of Y. If $U \subset X$ is open, then U is open in Y as well and $U = X \cap U$. Thus all the open subsets of X are open in the subsp. top.
If U is open in Y, is $U \cap X$ open in X?
If $\{\infty\} \notin U$, this is clear. Suppose $\{\infty\} \in U$. Then $U = Y - K$ where $K \subset X$ is compact. Then

$$U \cap X = (Y - K) \cap X = X - K.$$

Since X is Hausdorff, K is closed. Hence $X - K$ is open in X.

Now we will show that Y is cpt and H.

Compactness: Let $\mathcal{U}$ be an open cover of Y. Let $\infty \in U \in \mathcal{U}$ and $U = Y - K$, where K is compact. Let $\mathcal{U}' = \mathcal{U} - \{U\}$. Then $\mathcal{U}'$ is an open cover of K. Let $\mathcal{U}'' \subset \mathcal{U}'$ be a finite cover of K. Then $\mathcal{U}'' \cup \{U\}$ is a finite subcover of Y.

Hausdorff: Let $x, y \in Y$. If $x, y \in X$ we can separate them with open subsets of X. Suppose $x \in X$ and $y = \infty$. By loc. cpt. at x $\exists$ nbhd U s.t. $x \in U \subset K \subset X$, where K is compact. Let $V = Y - K$. Then $x \in U$, $\infty \in V$, $U \cap V = \emptyset$ and U and V are open.

Note: If X had been cpt. Then ∞ would be an isolated pt. in Y and $\bar{X} \neq Y$.

Next we show $\bar{X} = Y$. Let V be any nbhd of ∞. Then $V = Y - K$ for $K \subset X$ cpt. Since X is not cpt $K \neq X$. Thus $V \cap X \neq \emptyset$. Thus $\infty \in \bar{X}$, hence $\bar{X} = Y$.

(b) Let $Y = X \cup \{\infty\}$ be a one-pt-compactification of X. Since Y is Hausdorff (by definition) and $X \subset Y$, we know that X is Hausdorff.

Let $x \in X$. $\exists$ disjoint open sets U, V with $x \in U$ and $\infty \in V$. Let $C = Y - V$. Then C is cpt. and we have

$$x \in U \subset C \subset X.$$

Thus, X is loc. cpt.

(c) To show uniqueness suppose $Y = X \cup \{\infty\}$ and $Y' = X \cup \{\infty'\}$ are two 1-pt-compactifications of X. Let $h: Y \to Y'$ be given by

$$h(x) = \begin{cases} x & \text{if } x \in X \\ \infty' & \text{if } x = \infty. \end{cases}$$

Clearly h is a bijection. Let $U \subset Y$ be open. We claim $h(U) \subset Y'$ is open.

Case i: $\infty \notin U$. Then $h(U) = U \subset X \subset Y'$. Since X has the subsp. top and X is open in Y' (one pt sets are closed in H. sp's) we have U is open in $X \Rightarrow$ it is open in Y'!

Case ii: $\infty \in U$. Let $C = Y - U$. It is compact in X. Thus it is cpt. in Y and Y'. ($h(C) = C \subset X \subset Y'$.) Hence C is closed in Y'. Thus $h(U) = Y' - h(C)$ is open in Y'.

Hence h^{-1} is cont. Similarly h is cont. Thus, h is a homeo.

Thm (29.2) Let X be a Hausdorff sp. Then X is loc. cpt. iff $\forall x \in X$ and $\forall$ open nbhd U of x, $\exists$ an open nbhd V of x s.t. $\bar{V} \subset U$, and $\bar{V}$ is cpt. (In other words iff $\exists$ a cpt nbhd of x within every nbhd of X.)

Pf X is a nbhd of x, so the new condition implies $\exists$ a cpt nbhd of x. Hence X is loc cpt.

Suppose X is loc. cpt. Let $x \in U \subset X$, U open. Let Y be the 1-pt comp.... of X. Let $K = Y - U$. Thus K is a closed subset of a compact sp (Y), so K is cpt in Y.

By Lemma 26.4 (pg 166) since K is a cpt subsp of a H. sp and $x \notin K$, $\exists$ open, disjoint set ~~U~~, W with $x \in V$ and $K \subset W$. Then $\bar{V}$ is cpt in Y. No lim pt of V can be in W. Hence $\bar{V} \cap K = \phi$ so $\bar{V} \subset U$. $\bar{V}$ is also cpt in X, so we are done $\square$

Fact A A closed subset C of a loc. cpt. sp. X is loc. cpt. in the subsp. top.

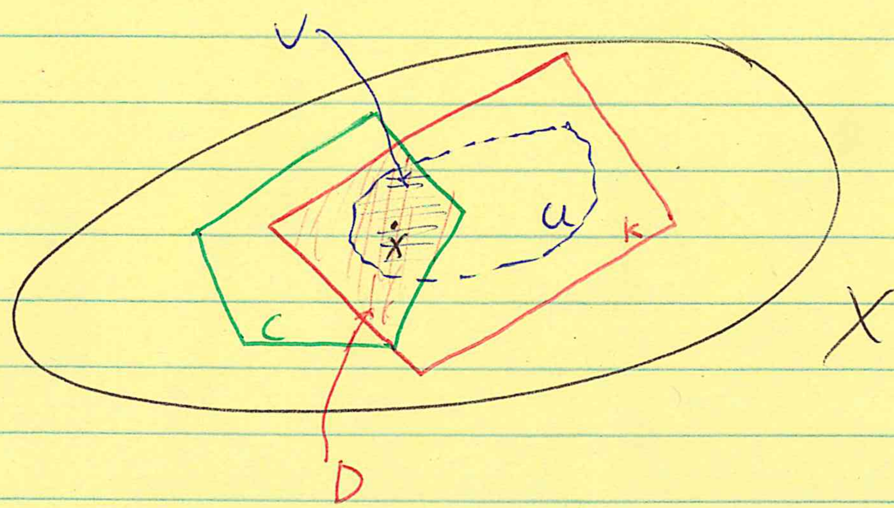
Pf Let $x \in C$. $\exists$ subsets of X, U open, K cpt with

$$x \in U \subset K,$$

since X is loc. cpt. Let $V = C \cap U$ and $D = C \cap K$. Then

$$x \in V \subset D \subset C.$$

Clearly V is open in the subsp. top of C. We claim D is cpt in C. We know D is closed in the subsp top of K. Hence D is cpt in C. Any open covering of D in K cooresponds to an open covering of D in X and then this cooresponds to an open covering of D in C. Thus, D is cpt in C. (See Lemma 26.1 on pages 164-5 for details.) $\boxed{\star}$



Fact B An open subset U of a loc. cpt. H. sp. X is loc. cpt. in the subsp. top.

Pf Let $x \in U$. By Thm 29.2, since U is a nbhd of X, $\exists V$ open in X with $x \in V$, $\bar{V} \subset U$, $\bar{V}$ cpt in X.

Now since V is open in X and $V \subset U$, V is open in U.

Since $\bar{V}$ is cpt in X and $\bar{V} \subset U$, an open set, we can show $\bar{V}$ is cpt in subsp. top. of U. Pf Let $\mathcal{A}$ be an open cover of $\bar{V}$ wrt U. But each member of $\mathcal{A}$ is also open in X. Thus $\exists$ a finite subcover.

Thus, U is loc. cpt. $\square$

Fact C If X is loc. cpt. H. sp., then X is homeo to an open subset of a cpt. H. sp.

Pf The copy of X in its 1 pt comp... is open: $X = Y - \{\infty\}$. $\square$

Fact D An open subset U of a cpt H sp. Y is loc. cpt. H.

Pf We know U is H. ~~Since~~ By Fact B, and that cpt $\Rightarrow$ loc. cpt, we have U is loc. cpt. $\blacksquare$