§30 The Countability Axioms

First countable (1)
Second countable (2)
Lindelöf (L)
Seperable (S)

All are preserved by homeomorphisms.

Def A top. sp. is second countable if it has a countable basis.

Ex $\mathbb{R}^n$ is 2 ct. Use balls with rational radii and centers with rational coordinates.

$\mathbb{R}^\omega$ with prod. top is 2 ct (Thm 30.2)

$\mathbb{R}^\omega$ with unif. top is not 2^{ct}. **Pf** Let A = seq's of 0's and 1's. A is uncountable. $\forall x \in A$ let $B_x = B_{\bar\rho}(x, \frac{1}{2})$. Now we have an unct. collection of disjoint open sets. Each must contain a distinct member of any given basis.

1 ct is a "local" version of 2 ct.

Def Let $x \in X$. A nbhd basis for x is any collection of nbhds of x, $\mathcal{U}$, s.t. for any open set V containing x, $\exists U \in \mathcal{U}$ s.t. $U \subset V$.

Ex Obviously $\mathcal{U}$ = all open sets containing x would work as a nbhd basis, but it is handy to find smaller ones. For $x \in \mathbb{R}^n$ the set of all open balls with center x is a nbhd basis. The set of all open balls with center x and rational radius is a countable nbhd basis. Even the set $\{B(x, \frac{1}{n}) \mid n = 1, 2, 3 \dots\}$ is a nbhd basis.

Def A top sp is <u>first countable</u> if every point has a countable nbhd basis.

Ex $\mathbb{R}^n$ is 1 ct. Any set X with the discrete top is 1 ct. Any metric sp is 1 ct.
$\mathbb{R}_\ell$ is 1 ct but not 2 ct. **Pf** $\{[x, x+\frac{1}{n})\}_{n=1}^{\infty}$ is a countable nbhd basis for any $x \in \mathbb{R}_\ell$. Now, suppose $\mathcal{B}$ is a basis for $\mathbb{R}_\ell$. $\forall x \in \mathbb{R}_\ell$, $\exists B_x \in \mathcal{B}$ s.t.

$$x \in B_x \subset [x, x+1).$$

If $x \neq y$, then B_x cannot equal B_y, since if $y > x$, $B_y \subset [y, y+1)$ and so $x \notin B_y$. Thus $\mathcal{B}$ is uncountable.

Fact 2 ct $\Rightarrow$ 1 ct. (Obvious.)

Facts (Thm 30.2)

Subsp's of 1 ct sp's are 1 ct.
Subsp's of 2 ct sp's are 2 ct
Countable products of 1 ct sp's are 1 ct.
Countable products of 2 ct sp's are 2 ct.

Pf Easy. See textbook. This fails for uncountable products.

Ex $\mathbb{R}^\omega_{prod}$ is 1^{st} ct and 2^{nd} ct, as is $\mathbb{R}^\infty$ as a subsp.

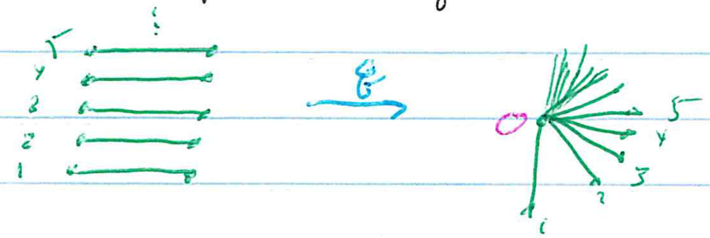
Fact (Exercise #12)

The cont. open image of a 1 ct sp is 1 ct.
The cont. open image of a 2 ct sp is 2 ct.

Ex The cont. image of a 1 ct sp need not be 1 ct.
$\mathbb{R}^\omega_{box}$ is not 1 ct. **Pf** Suppose $\mathcal{B} = \{B_1, B_2, \ldots\}$ is a countable basis for 0. For each $B_i \in \mathcal{B}$, $\exists$ a basis member $B_i' = \prod_{j=1}^{\infty} U_{ij}$ s.t. $0 \in B_i' \subset B_i$.
Then $\mathcal{B}' = \{B_1', \ldots\}$ is a countable basis for 0.
Let $V_{ii} \subset \mathbb{R}$ be open, contain 0, and be a proper subset of U_{ii}. Let $V = \prod V_i$. Then V is open in $\mathbb{R}^\omega_{box}$, $0 \in V$, but V does not contain any member of $\mathcal{B}'$.

Finally $\mathbb{R}^\omega_{discrete}$ is 1 ct and $id: \mathbb{R}^\omega_{dis} \to \mathbb{R}^\omega_{box}$ is continuous and onto. $\square$

Ex The cont. image of a 2ct sp need not be 2ct. Let $X = \mathbb{N} \times [0,1]$. Define $(n,x) \sim (m,y)$ iff $x=y=0$. Let $Z = X/\sim$ and let $q : X \to Z$ be the corresponding quotient map. Thus q is cont. and onto.



Since $\mathbb{N}$ and $[0,1]$ are 2ct, X is 2ct. We claim Z is not 1ct at $O = \{(n,0) \mid n \in \mathbb{N}\} \in Z$.

The saturated open subset of X are unions of sets of the form $\{n, U\}$, $U \subset (0,1]$, open and sets of the form

$$\bigcup_{n=1}^{\infty} (n, U_n)$$

where $U_n \subset [0,1]$ is open and $0 \in U_n$. Suppose

$$\mathcal{B} = \{B_1, B_2, B_3, \ldots\}$$

is a ct'able nbhd basis of O. Then each B_i contains a subset of the form

$$q\left(\bigcup_{n=1}^{\infty} (n, [0, \varepsilon_{ni}))\right) = O \cup \bigcup I_{ni}, \; i = 1,2,3\ldots$$

where I_n is the image of $(0, \varepsilon_n)$ in Z. Let $U = O \cup \bigcup_{i=1}^{\infty} (0, \frac{\varepsilon_{ii}}{2})$. Then no $B_i \subset U$. $\boxed{\cdot}$

Source Exercise 16B.1 in Willard's Gen. Top.

Def A top. sp. is Lindelöf if every open cover has a ct'able subcover.

Rmk Obviously Cpt $\Rightarrow$ Lin. But do not confuse Lin with countable compactness (see #4 in §28).

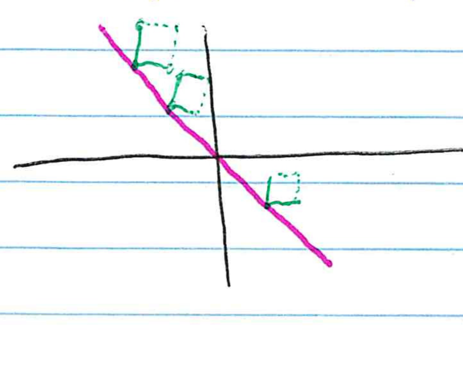
Ex $\mathbb{R}$ is Lin. **Pf** Let $\mathcal{A}$ be an open cover of $\mathbb{R}$. For each $n \in \mathbb{Z}$ let $\mathcal{A}_n$ be a finite subcover of $[n, n+1]$. Then $\bigcup \mathcal{A}_n$ is a countable subcover of $\mathbb{R}$. $\mathbb{R}^n$ and subspaces of $\mathbb{R}^n$ are Lin.

Ex $\mathbb{R}_\ell$ is Lin, $\mathbb{R}_\ell^2$ is not. The first claim is hard and we postpone its proof. Here we prove that $\mathbb{R}_\ell^2$ is not Lin. Let $U = \mathbb{R}_\ell^2 - \{(x,-x) \mid x \in \mathbb{R}_\ell\}$ and $B_x = [x, x+1) \times [-x, -x+1)$, $\forall x \in \mathbb{R}_\ell$.

Now U is open in $\mathbb{R}_\ell^2$ since any set open in $\mathbb{R}^2$ is open in $\mathbb{R}_\ell^2$. Clearly each B_x is open in $\mathbb{R}_\ell^2$. Thus,

$$\mathcal{A} = \{U\} \cup \{B_x \mid x \in \mathbb{R}_\ell\}$$

is an open cover of $\mathbb{R}_\ell^2$. Each set in $\mathcal{A}$ contains a pt of $\mathbb{R}_\ell^2$ that is not in any of member of $\mathcal{A}$. $\nexists$ any subcover. $\square$



Fact But Lin × Cpt is Lin. See Exercise #14.

Fact Subsp's of a Lin. sp need not be Lin. See Example #15.

Def A top sp is **separable** if it has a ct. dense subset.

Ex $\mathbb{R}^n$, $\mathbb{R}_\ell$, polynomials with rational coeff's are a ct dense subset of the sp. of analytic functions.

Fact 2 ct $\Rightarrow$ Lin and Sep. (Thm 30.3)

Fact For metric space Lin $\Rightarrow$ 2 ct and Sep $\Rightarrow$ 2 ct.
Thus for metric spaces $2 \Leftrightarrow L \Leftrightarrow S$.

Ex $\mathbb{R}^\omega_{\text{uniform}}$ is not Lin or Sep since it is not 2 ct.

Fact We can finially show $\mathbb{R}_\ell$ is not a metriz sp. (Recall Lecture 1). Since $\mathbb{Q}$ is dense in $\mathbb{R}_\ell$, $\mathbb{R}_\ell$ is Sep. But it is not 2 ct and hence it cannot be a metric sp.

Fact $\mathbb{R}_\ell$ is Lindelöf. This is Example #3 in the textbook.

Pf Let $\mathcal{A}$ be an open cover of $\mathbb{R}_\ell$. $\forall x \in \mathbb{R}_\ell$ $\exists A_x \in \mathcal{A}$ and $\varepsilon_x > 0$ s.t. $[x, x+\varepsilon_x) \subset A_x$. Let

$$\mathcal{B} = \{ [x, x+\varepsilon_x) \mid x \in \mathbb{R}_\ell \}.$$

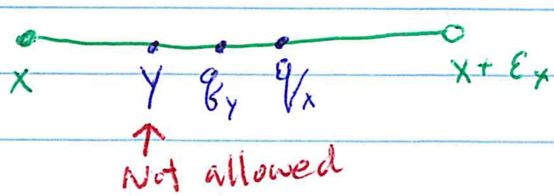
It is an open cover of $\mathbb{R}_\ell$. If we can find a ct subcover $\mathcal{B}' = \{ [x_i, \varepsilon_{x_i}) \mid i = 1, 2, 3, \dots \} \subset \mathcal{B}$, then $\mathcal{A}' = \{ A_{x_i} \mid i = 1, 2, 3 \dots \}$ will be a ct subcover of $\mathcal{A}$.

Define $C = \bigcup_{x \in \mathbb{R}_\ell} (x, x+\varepsilon_x)$. If $C = \mathbb{R}_\ell$ then

we are done since then $\{ (x, x+\varepsilon_x) \mid x \in \mathbb{R} \}$ is an open cover of $\mathbb{R}$ in the usual top and so it has a ct subcover, $\{ (x_i, x_i + \varepsilon_{x_i}) \mid i = 1, 2, 3 \dots \}$. But then $\mathcal{B}' = \{ [x_i, x_i + \varepsilon_{x_i}) \mid i = 1, 2 \dots \}$ is a ct subcover of $\mathcal{B}$.

Suppose $C \neq \mathbb{R}_\ell$. We claim $\mathbb{R}_\ell - C$ is ct. This will allow us to "patch up the holes!"

Pf $\forall x \in \mathbb{R}_\ell$, $\exists q_x \in [x, x+\varepsilon_x) \cap \mathbb{Q}$. If $x, y \in \mathbb{R}_\ell - C$ and $x < y$, then $q_x < q_y$; otherwise $y \in (x, x+\varepsilon_x) \subset C$.



Thus, we have constructed an injective map

$$x \mapsto q_x$$

of $\mathbb{R} - C \to \mathbb{Q}$. Hence $\mathbb{R} - C$ is ct'able.

Let $\mathcal{B}' = \{ [x, x+\varepsilon_x) \mid x \in \mathbb{R} - C \}$. It is a ct'able subcollection of $\mathcal{B}$ that covers $\mathbb{R} - C$.

We claim $\exists$ a ct'able subcollection of $\mathcal{B}$, call it $\mathcal{B}''$, that covers C. Since $\{ (x, x+\varepsilon_x) \mid x \in \mathbb{R}_\ell \}$ is an open cover of C in the usual top. of $\mathbb{R}$, $\exists$ a ct'able subcover $\{ (x_j, x_j + \varepsilon_j) \mid j = 1, 2, \ldots \}$. Then $\mathcal{B}'' = \{ [x_j, x_j + \varepsilon_{x_j}) \mid j = 1, 2, 3 \ldots \}$ covers C.

Then $\mathcal{B}' \cup \mathcal{B}'' \subset \mathcal{B}$ is a ct subcollection that covers $\mathbb{R}_\ell$. $\boxplus$