§31 The Separation Axioms

Def A top. sp. X is <u>regular</u> if

① one pt sets are closed, and

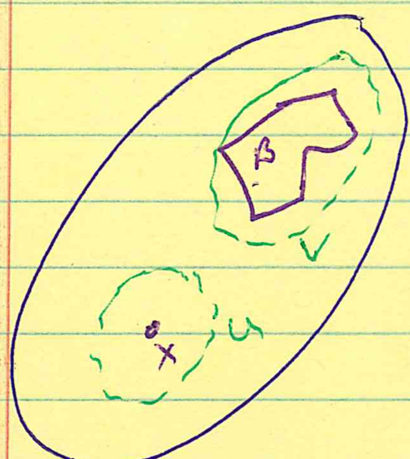
② pts and closed sets can be separated:
$\forall x, B$, $x \notin B$, B closed, $\exists U, V$ open disjoint with $x \in U$, $B \subset V$.

Def A top sp X is <u>normal</u> if

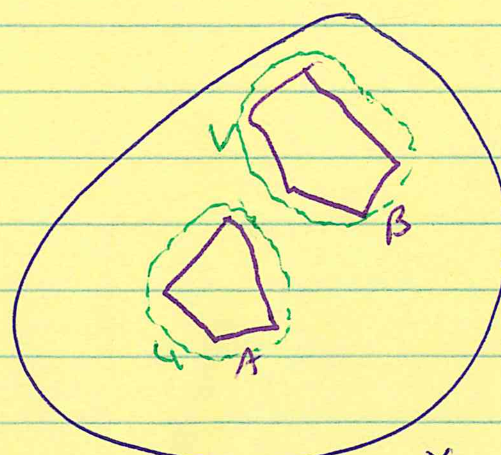
① one pt sets are closed, and

② closed sets can be separated.

Fact Normal $\Rightarrow$ Regular $\Rightarrow$ Hausdorff.
The reverse implications are false.



X reg.



X normal

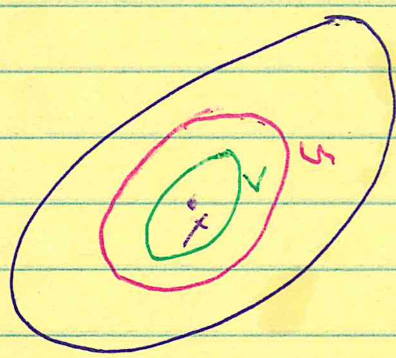
Ex In §32 we will show all metric spaces are normal.
$\mathbb{R}_K$ is Hausdorff but not reg. (See Example 1).
We will show $\mathbb{R}^2_\ell$ is reg. but not normal.

The next lemma gives alternative definitions

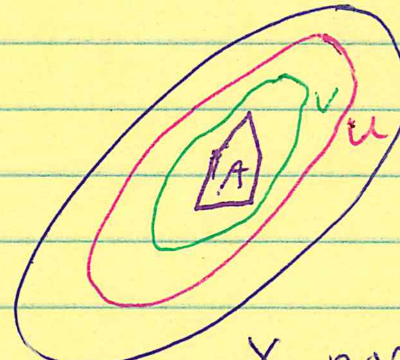
Lemma 31.1 Let X be a top. sp. in which one pt sets are closed

ⓐ X is reg. iff given any $x \in X$ and nbhd U of x, $\exists$ a nbhd V of x with $\bar{V} \subset U$.

ⓑ X is normal iff given any closed set A and open set U with $A \subset U$, $\exists$ V open with $A \subset V$, $\bar{V} \subset U$.



X reg



X normal

Pf of ($\Rightarrow$) of ⓐ (You can read the rest of the proof on your own) Let X be reg with $x \in U \subset X$, U open. Let $B = X - U$, B is closed and $x \notin B$. $\exists$ disjoint open sets V, W s.t. $x \in V$, $B \subset W$. This insures $\bar{V}$ is disjoint from B since if y is a limit pt of V every nbhd of y meets V, but W does not meet V, so $y \notin B$. Thus $\bar{V} \subset U$. $\boxed{\cdot}$

Thm (31.2) (a_1) A subsp. of a Hausdorff sp. is Haus.

(a_2) Products of H. sp's are Haus.

(b_1) A subsp of a reg. sp. is reg.

(b_2) Products of reg sp's are reg.

Pf (a_1) is Exercise 12 in §17 and is easy.

(a_2) the finite case is Exercise 11 in §17. The general case is not much harder. See proof in textbook.

(b_1) Let Y be a subsp of a reg sp. X. One pt sets in X are closed and hence they are closed in any subsp. Let $x \in Y$, $B \subset Y$, $x \notin B$, with B closed in Y. By Thm 17.4 $B = cl_X(B) \cap Y$. $x \notin cl_X(B)$ since $x \notin B$ and $x \notin X - Y$.

By the reg. of X $\exists$ U, V open disjoint sets in X with $x \in U$, $cl_X(B) \subset V$. Let $U' = U \cap Y$ and $V' = V \cap Y$. These separate x and B in Y. Thus Y is reg. $\square$

(b₂) We will use Lemma 31.1a. Let $X = \prod X_\alpha$ where each X_α is reg. Since X is Haus. one pt sets are closed in X with prod. top.

Let $x = (x_\alpha) \in X$ and U be a nbhd of x. Let $V = \prod V_\alpha$ be a basis member with $x \in V \subset U$. (Recall only finitely many V_α are not X_α.)

For each α choose W_α open in X_α s.t.

$$x_\alpha \in W_\alpha, \ \overline{W}_\alpha \subset V_\alpha, \quad \text{(Lemma 31.1a)}$$

but if $V_\alpha = X_\alpha$, we use $W_\alpha = X_\alpha$. Let $W = \prod W_\alpha$. It is an open nbhd of x.

By Thm 19.5 $\overline{W} \subset \prod \overline{W}_\alpha$. Thus $\overline{W} \subset \prod V_\alpha \subset U$. By Lemma 31.1a, X is reg. $\square$

However, a subspace of a normal sp. need not be normal. (See Example 1 in §32.) But a closed subsp. of a normal sp. is normal. (See Exercise 1 in §32.)

The product of two normal spaces need not be normal. We will show that $\mathbb{R}_\ell$ is normal, but $\mathbb{R}_\ell^2$ is not.

$\mathbb{R}_\ell$ is normal. (Example 2 is textbook.)

Pf It is easy to show one pt sets are closed since $(-\infty, a) \cup (a, \infty)$ is open.

Let A and B be disjoint closed subsets in $\mathbb{R}_\ell$. $\forall a \in A$ $\exists$ a basis member $[a, x_a)$ disjoint from B since $X - B$ is open. Likewise $\forall b \in B$ $\exists$ $[b, x_b)$ disjoint from A. Let

$$U = \bigcup_{a \in A} [a, x_a) \quad \text{and} \quad V = \bigcup_{b \in B} [b, x_b)$$

Clearly they are open and $A \subset U$, $B \subset V$.

If $U \cap V \neq \emptyset$ $\exists$ $\overset{\in A}{a}, \overset{\in B}{b}$ s.t. $[a, x_a) \cap [b, x_b) \neq \emptyset$. But then $b \in (a, x_a)$ or $a \in [b, x_b)$ which is impossible since

$$b \in B \text{ and } [a, x_a) \cap B = \emptyset, \text{ and}$$

$$a \in A \text{ and } [b, x_b) \cap A = \emptyset.$$

Thus $U \cap V = \emptyset$ and so $\mathbb{R}_\ell$ is normal. $\blacksquare$

$\mathbb{R}_\ell^2$ is not normal (Example 3 in textbook)

Pf We suppose $\mathbb{R}_\ell^2$ is normal and find a contradiction. Here is an outline of the plan.

- Let $L = \{(x,-x) \mid x \in \mathbb{R}_\ell\} \subset \mathbb{R}_\ell^2$. It is closed and has the discrete top. as a subsp.

- Let $D = \mathbb{Q}^2 \subset \mathbb{R}_\ell^2$. It is dense, $\overline{D} = \mathbb{R}_\ell^2$.

- Construct a map $\theta : \mathcal{P}(L) \to \mathcal{P}(D)$ that is one-to-one. Here we will use normality.

- Construct a one-to-one map $\psi : \mathcal{P}(D) \to L$. Thus, $\psi \circ \theta : \mathcal{P}(L) \to L$ is one-to-one. But this contradicts Thm 7.8 (pg 50) on cardinality.

We proceed. Let $A \subset L$. It is closed in L and hence in $\mathbb{R}_\ell^2$. The same is true for $L - A$. Assume $A \neq \emptyset$ or L. By normality $\exists$ disjoint open sets U_A and V_{L-A} s.t.

$$A \subset U_A \text{ and } L - A \subset V_{L-A}.$$

Define $\theta : \mathcal{P}(L) \to \mathcal{P}(D)$ by

$$\theta(A) = U_A \cap D \quad \text{for } A \neq \emptyset, L$$
$$\theta(\emptyset) = \emptyset,$$
$$\theta(L) = D.$$

To show θ is injective let $A, B \subset L$, $A \neq B$ and niether are $\emptyset$ or L. We need to show $\theta(A) \neq \theta(B)$. Since $A \neq B$ $\exists x \in A - B$ or $B - A$. Wlog let $x \in A - B$.

Since $x \notin B$, $x \in L - B$. Thus $x \in A \subset U_A$ and $x \in L - B \subset V_{L-B}$. Thus $U_A \cap V_{L-B} \neq \emptyset$. Since it is open and D is dence, $\exists d \in D$ in $U_A \cap V_{L-B}$. Therefore $d \notin U_B$. Hence

$$D \cap U_A \neq D \cap U_B.$$

Thus

$$\theta(A) \neq \theta(B).$$

Now we define $\Psi : \mathcal{P}(D) \to L$. We know there is a bijection between D and $\mathbb{N}$, Hence $\exists$ a bij. between $\mathcal{P}(D)$ and $\mathcal{P}(\mathbb{N})$. (Why?) There is an obvious bij between L and $\mathbb{R}$. Here is a one-to-one map from $\mathcal{P}(\mathbb{N})$ into $\mathbb{R}$.

Let $S \subset \mathbb{N}$. Define $\rho(S) = \sum_{i=1}^{\infty} \frac{a_i}{10^i}$ where

$$a_i = \begin{cases} 0 & \text{if } i \notin S \\ 1 & \text{if } i \in S. \end{cases}$$

We give some examples. $\rho(\{1,2,4\}) = 0.1101$, $\rho(\text{evens}) = 0.01010101\ldots$, $\rho(\mathbb{N}) = 0.1111\ldots$, $\rho(\emptyset) = 0.0$. Since two real numbers, without repeating 9's, are equal iff they have the same decimal expansion, ρ is one-to-one.

Now we have

$$\mathcal{P}(L) \xrightarrow{\theta} \overbrace{\mathcal{P}(D) \longleftrightarrow \mathcal{P}(\mathbb{N}) \xrightarrow{\rho} \mathbb{R} \longleftrightarrow L}^{\psi}$$

is one-to-one.