§ 32 Normal Spaces

Thm 32.1 Reg and 2nd ct $\Rightarrow$ Normal.

Pf is hard. We will do this.

Thm 32.2 Metric $\Rightarrow$ Normal.*

Pf is easy. See textbook.

Thm 32.3 Compact and Hausdorff $\Rightarrow$ Normal.

Pf is easy. See textbook.

Thm 32.4 Well ordered in order top $\Rightarrow$ Normal.

Pf is of medium difficulty. Read on your own.

Exercise #4 Reg. and Lin. $\Rightarrow$ Normal.

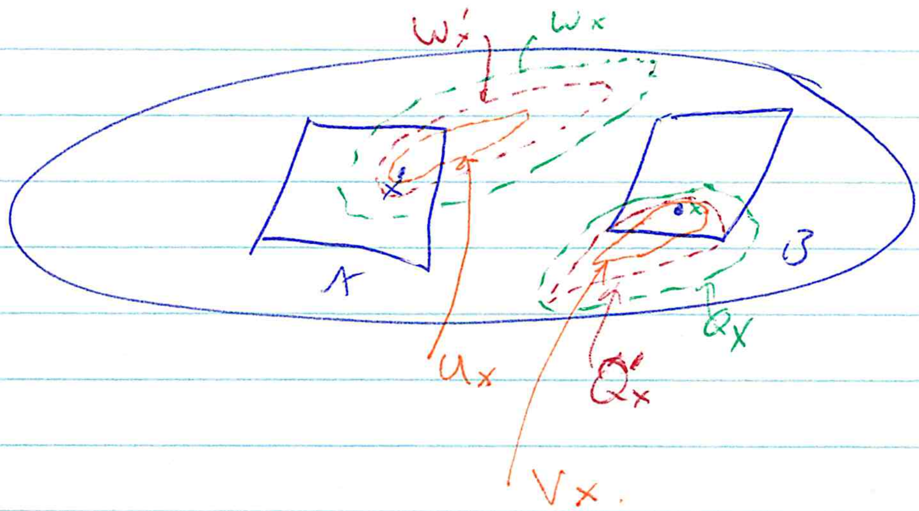
Pf: This is homework. Similar to 32.1

* Metric spaces are completely normal (§32 7(c)) and prefectly normal (§33 #6).

Pf of 32.1 Let X be a reg. sp. with ct. basis $\mathcal{B}$. Let A and B be disjoint closed sets in X.

$\forall x \in A$, $\exists\ W_x$, a nbhd of x disjoint from B.
$\forall x \in B$, $\exists\ Q_x$, a nbhd of x disjoint from A.
Now $A \subset \bigcup W_x$ and $B \subset \bigcup Q_x$, and while these unions are open they need not be disjoint.

By Lemma 31.1a $\exists\, W'_x$, a nbhd of x with $\overline{W'_x} \subset W_x$ $\forall x \in A$ and $\exists\ Q'_x$, a nbhd of x with $\overline{Q'_x} \subset Q_x$.
Further, $\exists$ a basis member U_x s.t. $x \in U_x \subset W'_x$, $\forall x \in A$, and $\exists$ a basis member V_x s.t. $x \in U_x \subset Q'_x$, $\forall x \in B$.



But U_x and V_x need not be disjoint. Now we use that $\mathcal{B}$ is ct. and enumerate

$$\{U_x \mid x \in A\} = \{U_1, U_2, U_3, \ldots\} \quad \text{and}$$

$$\{V_x \mid x \in B\} = \{V_1, V_2, V_3, \ldots\}.$$

For each n define $U_n' = U_n - \bigcup_{i=1}^{n} \overline{V}_i$ and

$$V_n' = V_n - \bigcup_{i=1}^{n} \overline{U}_i .$$

Let $U' = \bigcup_{n=1}^{\infty} U_n'$ and $V' = \bigcup_{n=1}^{\infty} V_n'$. They are open and $A \subset U'$ and $B \subset V'$. We claim they are disjoint.

$\boxed{n=1}$ $U_1' = U_1 - \overline{V}_1$ and $V_1' = V_1 - \overline{U}_1$.

Since $U_1' \cap V_1 = \phi$ and $V_1' \subset V_1$, we have $U_1' \cap V_1' = \phi$.

$\boxed{n=2}$ $U_2' = U_2 - (\overline{V}_1 \cup \overline{V}_2)$ and $V_2' = V_2 - (\overline{U}_1 \cup \overline{U}_2)$.

$U_2' \cap V_2 = \phi$ and $V_2' \subset V_2 \Rightarrow U_2' \cap V_2' = \phi$

$U_2' \cap V_1 = \phi$ and $V_1' \subset V_1 \Rightarrow U_2' \cap V_1' = \phi$

$V_2' \cap U_1 = \phi$, $U_1' \subset U_1 \Rightarrow V_2' \cap U_1' = \phi$

Hence $\bigcup_{n=1}^{2} U_n' \cap \bigcup_{n=1}^{3} V_2' = \phi$.

This works for any n.

$U_n' = U_n - (\bar{V}_1 \cup \bar{V}_2 \cup \cdots \cup \bar{V}_n)$ ad $V_n' = V_n - (\bar{U}_1 \cup \cdots \cup \bar{U}_n)$.

For $j \leq n$ we have:

$U_n' \cap V_j = \phi$ and $V_j' \subset V_j \Rightarrow U_n' \cap V_j' = \phi$.

Likewise $V_n' \cap U_j' = \phi$, $j = 1 \ldots n$.

If $U' \cap V' \neq \phi$, then $\exists\, i, j$ s.t. $U_i' \cap V_j' \neq \phi$.
But we have just shown this never hapens.

Thus we have found U', U', open disjoint with $A \subset U'$ and $B \subset V'$. Hence X is normal.