§34 Urysohn's Metrization Thm

Thm 34.1 Every 2^{nd} ct. reg. sp. is metrizable.

Rmk

$$\begin{array}{ccc} (\text{Reg.}, 2^{nd}\text{ ct}) & \overset{32.1}{\Longrightarrow} & \text{Normal} \Rightarrow \text{Reg.} \\ {\scriptstyle 34.1}\Downarrow & & \Uparrow{\scriptstyle 32.2} \\ & \text{Metrizable} & \end{array}$$

Thus for 2^{nd} ct. sp's we have Reg $\Leftrightarrow$ Normal $\Leftrightarrow$ Met.

But, Normal $\not\Rightarrow$ Metrizable (§34 Exercise #2)

And, H, 2^{nd} ct $\not\Rightarrow$ Metrizable (§34 Exercise #1 [$\mathbb{R}_K$])

Idea behind the proof of 34.1

Let X be reg. and 2^{nd} ct. We construct an embedding of X into $\mathbb{R}^\omega$. (Recall that $\mathbb{R}^\omega$ in prod. top. is metrizable, see Thm 20.5 (pg 125).) Then X is homeo to its image in $\mathbb{R}^\omega$. The image inherits the metric from $\mathbb{R}^\omega$ and is thus a metric space.

This is a common strategy. If you want to show a sp. X has a property P this is inherited by subspaces, embed X into a larger space with property P.

Now the proof! We define a map $F: X \to \mathbb{R}^\omega$ as $F(x) = (f_1(x), f_2(x), \dots)$, as follows. Let $\{B_i\}_{i=1}^\infty$ be a ctb basis for X.

For each pair of indicies for which $\overline{B}_n \subset B_m$ choose a cont. function $g_{m,n}: X \to [0,1]$ s.t.

$$g_{m,n}(\overline{B}_n) = \{1\} \text{ and } g_{mn}(X - B_m) = \{0\}.$$

These exist by Urysohn's Lemma. (That such pairs B_m, B_n exist comes from regularity.)

The collection $\{g_{m,n}\}$ is indexed over some subset of $\mathbb{N} \times \mathbb{N}$ and is clearly countable. Relabel it as $\{f_i\}_{i=1}^\infty$. Let

$$F(x) = (f_1(x), f_2(x), f_3(x), \dots).$$

F is cont. since each f_i is cont.

We claim F is one-to-one. Suppose $x, y \in X$, $x \neq y$. Let U be a nbhd of x that does not contain y. $\exists\, m, n$ s.t. $x \in B_n$, $\overline{B}_n \subset B_m \subset U$, by regularity using Lemma 31.1a. Let $f_k = g_{m,n}$. Then

$$f_k(x) = 1 \text{ and } f_k(y) = 0.$$

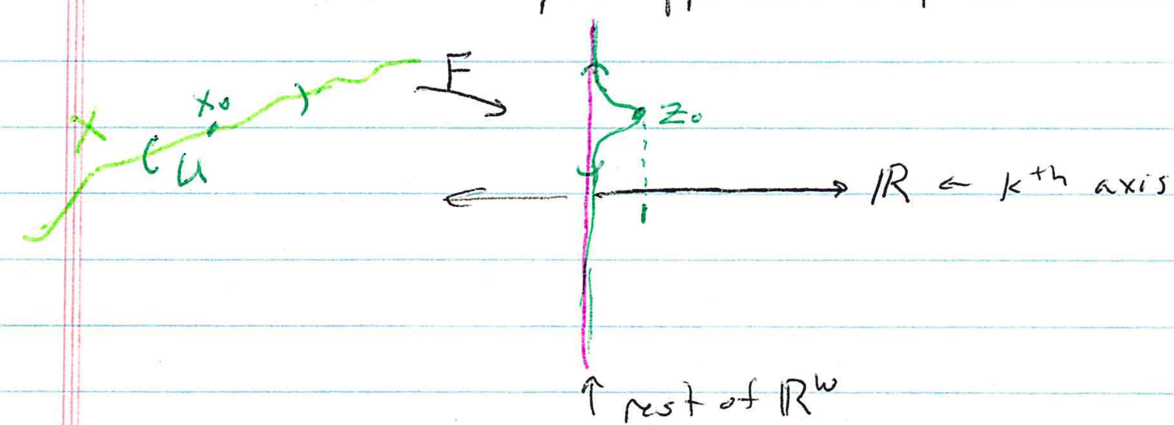
Hence $F(x) \neq F(y)$. So, F is one-to-one.

Let $Z = F(X) \subset \mathbb{R}^\omega$. F^{-1} is well defined on Z. We show it is cont. by showing F is an open map (i.e. $F = (F^{-1})^{-1}$ takes open sets to open sets $\Rightarrow F^{-1}$ cont.)

Let $U \subset X$ be open. We must show $F(U)$ is open in Z with the subsp top. Let $z_0 \in F(U) \subset Z$. We will find a nbhd W of x, that is $x \in W \subset Z$, and W is open in the subsp. top. of Z. This will be nontrivial.

Let $x_0 = F^{-1}(z_0) \in U$. Let k be s.t. $f_k(x_0) = 1$ and $f_k(X - U) = \{0\}$. (k exists since $\exists$ basis members B_n, B_m with $x \in B_n$, $\bar{B}_n \subset B_m \subset U$.)

Let $V = \pi_k^{-1}(0, \infty) = \{(x_1, x_2, \dots, x_k, \dots) \mid x_k > 0\} \subset \mathbb{R}^\omega$. It is the k^{th} open upper half "plane".



Let $W = V \cap Z$, which is open in the subsp top of Z.

Now $\pi_k(z_0) = f_k(x_0) = 1 > 0$. Thus $z_0 \in V$ so $z_0 \in W$.

It is only left to show that $W \subset F(U)$.

Let $z \in W$, and $x = F^{-1}(z)$. $\pi_k(z) = \pi_k(F(x)) = f_k(x)$.

Since $z \in W \subset V$ we know $\pi_k(z) > 0$. Thus $f_k(x) > 0$. Since f_k is zero outside U, it follows that $x \in U$. Hence $z \in F(U)$ and $W \subset F(U)$ since $z \in W$ was arbitrary.

Thus $F(U)$ is open and so F^{-1} is cont. Hence $F : X \to \mathbb{R}^\omega$ is an embedding so $F(X)$ is homeo to X. Since $F(X) \subset \mathbb{R}^\omega$ is metrizable, so is X.