§36 Imbeddings* of Manifolds

Def An *n-dimensional manifold*, or *n-manifold* is a second countable Hausdorff space where each pt has a nbhd homeomorphic to an open subset of $\mathbb{R}^n$.

Ex's $\mathbb{R}^n$, S^n, T^n, Klein bottle, ...

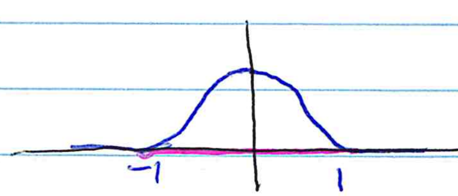
Goal Recall an *imbedding* of a top. sp. X into a top. sp. Y is a map $f: X \to Y$ that is one-to-one, continuous and $f^{-1}: f(X) \to X$ is cont. (Page 105)

Note: Many texts define imbeddings to by smooth; here we are using *topological imbeddings*.

Our goal is to prove that for any manifold M $\exists N$ s.t. M can be imbedded into $\mathbb{R}^N$.

Def If $f: X \to \mathbb{R}$, then the *support* of f is $\overline{f^{-1}(\mathbb{R} - \{0\})}$. The notation is *supp(f)*.

Ex Let $f(x) = \begin{cases} e^{-1/1-x^2} & x \in (-1,1) \\ 0 & \text{otherwise.} \end{cases}$ Then $\text{supp}(f) = [-1, 1]$.



*Can also be spelt, "embeddings".

Def Let $\mathcal{U} = \{U_1, U_2, \ldots, U_k\}$ be an open cover of a top. sp. X. A partition of unity wrt $\mathcal{U}$ is a family of cont. functions

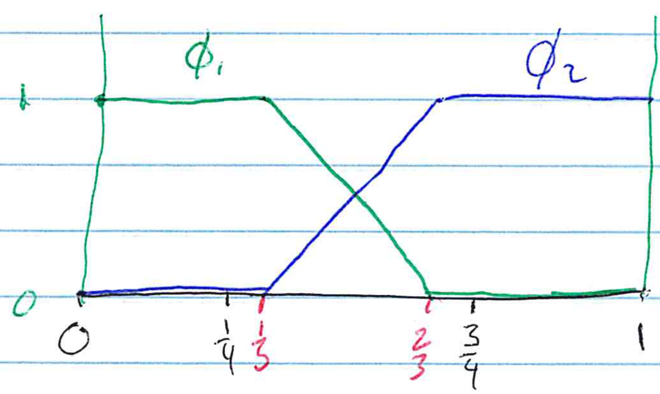
$$\phi_i : X \to [0,1], \quad i = 1, \ldots, k,$$

s.t. $\text{supp}(\phi_i) \subset U_i$, $i = 1..k$ and $\sum_{i=1}^{k} \phi_i(x) = 1 \ \forall x \in X$.

Ex Let $X = [0,1]$ with subsp. top. Let $\mathcal{U} = \{[0, \frac{3}{4}), (\frac{1}{4}, 1]\}$.
Let

$$\phi_1(x) = \begin{cases} 1 & x \in [0, \frac{1}{3}] \\ -3x+2 & x \in [\frac{1}{3}, \frac{2}{3}] \\ 0 & x \in [\frac{2}{3}, 1] \end{cases}$$

$$\phi_2(x) = \begin{cases} 0 & x \in [0, \frac{1}{3}] \\ 3x-1 & x \in [\frac{1}{3}, \frac{2}{3}] \\ 1 & x = [\frac{2}{3}, 1]. \end{cases}$$

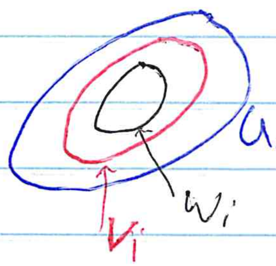


$\phi_1 + \phi_2 = 1$

$\text{supp}(\phi_1) = [0, \frac{2}{3}] \subset [0, \frac{3}{4})$, $\text{supp}(\phi_2) = [\frac{1}{3}, 1] \subset (\frac{1}{4}, 1]$.

Thm (36.1, Existence) Let $\mathcal{U} = \{U_1, \dots, U_k\}$ be a finite open cover of a normal sp. X. Then $\exists$ a part. of unity wrt $\mathcal{U}$.

Pf First we outline the main ideas. We will "shrink" the open U_i's to "smaller" open sets that still cover X. This will be done in two stages:



s.t. $\overline{W}_i \subset V_i$, $\overline{V}_i \subset U_i$.

Then apply Urysohn's lemma to $\overline{W}_i$ and $X - V_i$ to create a function f_i that is 1 on $\overline{W}_i$ and 0 on $X - V_i$. Then these are "rescaled" to get the sum $\sum f_i(x) = 1 \ \forall x \in X$.

Step 1 We prove $\exists \{V_1, \dots V_k\}$, an open cover of X s.t. $\overline{V}_i \subset U_i$. We use induction. Let

$$A = X - (U_2 \cup \cdots \cup U_k).$$

It is closed and $A \subset U_1$. By normality $\exists V_1$ open s.t. $A \subset V_1$, $\overline{V}_1 \subset U_1$ (Lemma 31.1b). Now $\{V_1, U_2, \dots, U_k\}$ covers X.

Suppose we have been able to repeat this process $n-1$ times and now have $V_1, V_2, \ldots, V_{n-1}$ open with $\overline{V}_i \subset U_i$, $i=1,\ldots,n-1$, and that

$$\{V_1, \ldots, V_{n-1}, U_n, \ldots, U_k\}$$

covers X. Let

$$A = X - (V_1 \cup V_2 \cup \cdots \cup V_{n-1}) - (U_{n+1}, \ldots, U_k).$$

Now A is closed and $A \subset U_n$. Again by normality $\exists\, V_n$ open s.t. $A \subset V_n$, $\overline{V}_n \subset U_n$.

We can thus continue until we have $\{V_1, \ldots, V_k\}$ a open cover of X with $\overline{V}_i \subset U_i$, $i=1\ldots k$.

Step 2 Do step 1 again and produce a new open cover of X $\{W_1, \ldots, W_k\}$ s.t. $\overline{W}_i \subset V_i$, $i=1\ldots k$.

By Urysohn's Lemma, for each $i=1\ldots k$, $\exists$ a cont. map

$$f_i : X \to [0,1]$$

s.t. $f_i(x)=1$ on $\overline{W}_i$ and $f_i(x)=0$ on $X-V_i$. Thus

$$\mathrm{supp}(f_i) = \overline{f_i^{-1}(\mathbb{R}-\{0\})} \subset \overline{V}_i \subset U_i.$$

Consider the sum $F(x) = \sum_{i=1}^{k} f_i(x)$. It can never be zero since each x is in some W_i. So, it takes values in $[1, k]$. Let $g_i(x) = f_i(x)/F(x)$. Then
$$G(x) = \sum g_i(x) = \frac{\sum f_i(x)}{\sum f_i(x)} = 1 \quad \forall x \in X.$$

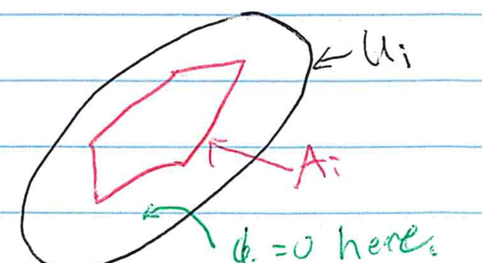
Thus $g_1, \dots, g_k$ is the desired partition of unity wrt to $\{U_1, \dots, U_k\}$. ∎

Thm (36.2) If X is a compact n-manifold, then $\exists N$ s.t. X can be embedded into $\mathbb{R}^N$.

Pf By Thm 32.3 X is normal. Thus Thm 36.1 can be applied. For each $x \in X$ $\exists$ open U_x, containing x that can be embedded into $\mathbb{R}^n$. Thus form an open cover of X. Let $\{U_1, \dots, U_k\}$ be a finite subcover. Let $g_i : U_i \to \mathbb{R}^n$ be an embedding for $i = 1 \dots k$. Let $\phi_i : U_i \to \mathbb{R}$, $i = 1 \dots k$, be a partition of unity wrt $\{U_1, \dots, U_k\}$.

Let $A_i := \text{supp}(\phi_i) \subset U_i$ for each i. Define $h_i : X \to \mathbb{R}^n$ by

$$h_i(x) = \begin{cases} \underbrace{\phi_i(x)}_{\text{scalar}} \underbrace{g_i(x)}_{\text{vector}} & x \in U_i \\ \underbrace{0}_{\text{vector}} & x \in X - A_i. \end{cases}$$



It is well defined since $\phi_i(x) = 0$ on $U_i - A_i$. h_i is cont. by Thm 18.3 (The pasting lemma.)

Now define $F : X \to \mathbb{R}^k \times \mathbb{R}^{kn}$ by

$$F(x) = (\underbrace{\phi_1(x), \phi_2(x), \dots, \phi_k(x)}_{\text{scalars}}, \underbrace{h_1(x), h_2(x), \dots h_k(x)}_{\text{vectors in } \mathbb{R}^n}).$$

We claim F is an imbedding. Cont. is clear (Thm 18.4). We will show that F is 1-to-1. Then by Thm 26.6, since X is compact and $\mathbb{R}^{k+kn}$ is Hausdorff, F^{-1} is cont. on $F(X)$.

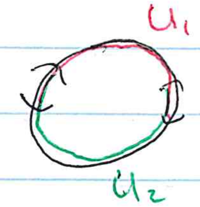
Suppose $F(x) = F(y)$. Then $\phi_i(x) = \phi_i(y)$ and $h_i(x) = h_i(y)$ for each $i = 1 \dots k$. Since $\sum \phi_i(x) = 1$ $\exists j$ s.t. $\phi_j(x) > 0$. Thus $x \in U_j$. Since $\phi_j(y) = \phi_j(x) > 0$, $y \in U_j$ also. So, at least x and y are neighbors. Next we'll show they are equal.

We claim that $g_j(x) = g_j(y)$. Since $g_j : U_j \to \mathbb{R}^n$ is an embedding, this means $x = y$.

We have $h_j(x) = h_j(y)$. Thus $\phi_j(x) g_j(x) = \phi_j(y) g_j(y)$ by definition of h_j. Since $\phi_j(x) = \phi_j(y) > 0$ we get $g_j(x) = g_j(x)$.

Thus, $x = y$. Hence F is one-to-one and we are done. $\square$

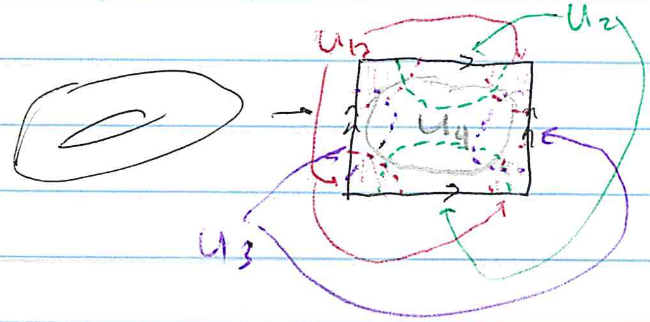
Some examples

S^1: 

$n=1, k=2$. Thus $\exists$ an embedding of S^1 into $\mathbb{R}^4$.

S^2 

$n=2, k=2$. Thus $\exists$ an embedding of S^2 into $\mathbb{R}^6$.

T^2 

$k=4$

$n=2$

Thus $\exists$ an embedding of T^2 into $\mathbb{R}^{4+4\cdot 2} = \mathbb{R}^{12}$.

K - the Klein bottle. We can create an open cover with $k=4$ just like we did for T^2. Thus, $\exists$ an embedding of K into $\mathbb{R}^{12}$.

Obviously, the ~~value~~ dimension of $\mathbb{R}^N$ that we find in this way is not optimal. There is a stronger result.

Thm (The Whitney Embedding Thm) Let M^n be an n-dimensional manifold. Then $\exists$ a smooth embedding

$$M^n \hookrightarrow \mathbb{R}^{2n+1}.$$

See Differential Topology by Guillemin & Pollack.