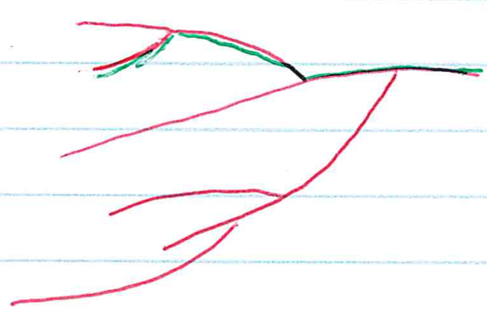
§37 # The Tychonoff Theorem

Here we will prove that the product of cpt spaces is cpt. We have done this for finite products (Thm 26.7). Recall the F.I.P. and Thm 26.9: cpt $\Leftrightarrow$ ($\mathcal{C}$ F.I.P. $\Rightarrow \bigcap \mathcal{C} \neq \emptyset$).

We will make use of Zorn's Lemma, which is a consequence of the Maximum Principle (§11). We review these briefly.

MP Let X be a set with a strict partial order. (Proper set inclusion of a collection of sets is a typical example of a s.p.o.) Then $\exists$ a maximal simply ordered subset. (It need not be unique.

Ex In the graph below $a < b$ iff there is a path from a to b that always moves left-to-right.



The green subset is a maximal simply ordered subset. If any other point is included it will no longer be simply ordered. (More examples are on the website.)

ZL Let X be a set with a strict partial order. If $\forall$ simply ordered subset of X, $\exists$ an upper bound in X, then X has a maximal member. (It need not be unique.

pf (Outline) By the MP X has a maximal simply ordered subset Y. The upper bound of Y is a maximal member of X. $\blacksquare$

We will be working with subsets of a set X, collections of subsets of X, and collections of collections of subsets of X. To help keep these straight we adopt the following convention.

Subsets of a given set X will be denoted by standard capital letters: $A, B, C, \ldots$

Collects of subsets of X will be denoted by script capital letters: $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$

Collects of collections of subsets of X will be denoted by bold capital letters: $\mathbf{A}, \mathbf{B}, \ldots$

The next two lemmas only concern sets, not top. spaces.

Lemma (37.1) Let X be a set. Let $\mathcal{A}$ be a collection of subsets of X having the F.I.P. Then $\exists$ a collection $\mathcal{D}$ of subsets of X s.t. $\mathcal{A} \subset \mathcal{D}$, $\mathcal{D}$ has the F.I.P., and $\mathcal{D}$ is maximal, meaning that no larger collection containing $\mathcal{D}$ has the F.I.P.

Pf Let $\mathbf{A}$ be the collection of all collections of subsets of X that contain $\mathcal{A}$ and have the F.I.P. Proper set inclusion gives a strict partial order on $\mathbf{A}$.

Let $\mathbf{B} \subset \mathbf{A}$ be simply ordered. We claim $\mathbf{B}$ has an upper bound in $\mathbf{A}$, namely $\mathcal{C} = \bigcup_{\mathcal{B} \in \mathbf{B}} \mathcal{B}$. Clearly $\mathcal{C}$ is an upper of $\mathbf{B}$; if $\mathcal{B} \in \mathbf{B}$, then $\mathcal{B} \subset \mathcal{C}$. But we need for $\mathcal{C} \in \mathbf{A}$. It is clear that $\mathcal{A} \subset \mathcal{C}$ since $\mathcal{A} \subset \mathcal{B}$ for all $\mathcal{B} \in \mathbf{B}$. It is left to show that $\mathcal{C}$ has the F.I.P.

Let $\{C_1, \dots, C_n\} \subset \mathcal{C}$. Each C_i is in some $\mathcal{B}_i \in \mathbf{B}$. We can write $C_i \in \mathcal{B}_i \in \mathbf{B}$, $i = 1, \dots, n$. Since $\mathbf{B}$ is simply ordered by proper set inclusion, the sub-collection $\{\mathcal{B}_1, \dots, \mathcal{B}_n\} \subset \mathbf{B}$ has a largest member, say $\mathcal{B}_k$. Then $\{C_1, \dots, C_n\} \subset \mathcal{B}_k$ and we know $\mathcal{B}_k$ has the F.I.P. Hence $\bigcap_{i=1}^{n} C_i \neq \emptyset$. Therefor $\mathcal{C} \in \mathbf{A}$.

By Zorn's Lemma $\mathbf{A}$ has a maximal member which is the collection $\mathcal{D}$ in the statement of this Lemma. $\square$

Lemma (37.2) Let X be a set. Let $\mathcal{D}$ be a collection of subsets of X that is maximal w.r.t. the F.I.P. Then the following hold

ⓐ Any finite intersection of elements of $\mathcal{D}$ is an element of $\mathcal{D}$.

ⓑ If A is a subset of X that meets every member of $\mathcal{D}$, then $A \in \mathcal{D}$.

Pf The proofs are easy. You can read them in the textbook.

Proof of the Tychonoff Theorem

Let $X = \prod_{\alpha \in J} X_\alpha$, where each X_α is cpt and X has the prod. top. Let $\mathcal{A}$ be a collection of closed subsets of X having the F.I.P. We will show that

$$\bigcap_{A \in \mathcal{A}} A \neq \emptyset.$$

It follows that X is compact.

Choose $\mathcal{D} \supset \mathcal{A}$ as in Lemma 37.1 Members of $\mathcal{D}$ need not be closed, but if we show that $\bigcap_{D \in \mathcal{D}} \overline{D} \neq \emptyset$ then $\bigcap_{A \in \mathcal{A}} A \neq \emptyset$.

$\forall \alpha \in J$ consider $\{\pi_\alpha(D) \mid D \in \mathcal{D}\}$. Each is a collection of subsets of X_α that has the F.I.P. Since X_α is compact

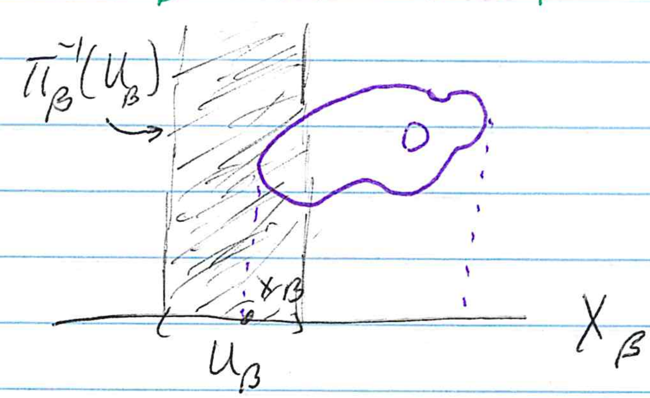
$$\bigcap_{D \in \mathcal{D}} \overline{\pi_\alpha(D)} \neq \phi.$$

$\forall \alpha \in J$ pick a point $x_\alpha \in \bigcap_{D \in \mathcal{D}} \overline{\pi_\alpha(D)}$. Let $\bar{x} = (x_\alpha)_{\alpha \in J}$.

We will show that $\bar{x} \in \overline{D}$, $\forall D \in \mathcal{D}$.

Recall that $\{\pi_\alpha^{-1}(U_\alpha) \mid \alpha \in J$ and U_α is open in $X_\alpha\}$ is a subbasis for the prod. top. on X. Let $\pi_\beta^{-1}(U_\beta)$ be a subbasis member that contains $\bar{x}$. Then $x_\beta \in U_\beta$. We claim $\pi_\beta^{-1}(U_\beta)$ meets every member of $\mathcal{D}$. Let $D \in \mathcal{D}$. Since $x_\beta \in \overline{\pi_\beta(D)}$ and U_β is open we have $U_\beta \cap \pi_\beta(D) \neq \phi$. Thus,

$$\pi_\beta^{-1}(U_\beta) \cap D \neq \phi.$$



By Lemma 37.2 (b) every subbasis member that contains $\bar{x}$ is in $\mathcal{D}$.

By Lemma 37.2 (a) every ~~bases~~ basis member that contains $\bar{x}$ is in $\mathcal{D}$.

Since $\mathcal{D}$ has the F.I.P. every basis member that contains $\bar{x}$ intersects every member of $\mathcal{D}$. Thus, $\bar{x} \in \bar{D}$ for every $D \in \mathcal{D}$. Thus,

$$\bar{x} \in \bigcap_{D \in \mathcal{D}} \bar{D} \subset \bigcap_{A \in \mathcal{A}} A \quad \text{and hence} \quad \bigcap_{A \in \mathcal{A}} A \neq \emptyset. \quad \boxtimes$$