§39 Local Finiteness

Def Let X be a top. sp. A collection of subsets $\mathcal{A}$ is locally finite if $\forall p \in X$, $\exists$ a nbhd of p that meets only finitely many members of $\mathcal{A}$. If $\mathcal{A}$ can be written as a countable union of subcollections $\mathcal{A}_n$, each of which is loc. f. then $\mathcal{A}$ is countably locally finite.

Ex $\{(n, n+2)\}_{n \in \mathbb{Z}}$ is a loc. f. open cover of $\mathbb{R}$.

Ex $\mathcal{A} = \{(-n, n)\}_{n \in \mathbb{N}}$ is a countably loc. f. open cover of $\mathbb{R}$. To see this let $\mathcal{A}_n = \{(-n, n)\}$. Note that these subcollections are not covers of $\mathbb{R}$.

Def Let $\mathcal{A}$ be a collection of subsets of a top. sp. X. Another collection $\mathcal{B}$ is said to be a refinement of $\mathcal{A}$ if $\forall B \in \mathcal{B}$, $\exists A \in \mathcal{A}$ s.t. $B \subset A$.

Ex $\{B(p, 1) \mid p \in \mathbb{R}^2\}$ is an open cover of $\mathbb{R}^2$.
$\{B(p, \frac{1}{2}) \mid p \in \mathbb{Q}^2\}$ is a refinement that is also an open cover of $\mathbb{R}^2$.

Lemma 39.1 Let $\mathcal{A}$ be a loc. f. collection of subsets of X. Then
ⓐ any subcollection or refinement is loc. f. (Obvious.),
ⓑ letting $\bar{\mathcal{A}} = \{\bar{A} \mid A \in \mathcal{A}\}$, then $\bar{\mathcal{A}}$ is loc. f., and
ⓒ $\overline{\cup \mathcal{A}} = \cup \bar{\mathcal{A}}$.

Pf (b) follows from the fact that any open set meeting $\bar{A}$ meets A.

(c) We already know that $\bigcup \bar{\mathcal{A}} \subset \overline{\bigcup \mathcal{A}}$ in general from exercise 6c in §17. (pg 101). Recall the reverse inclusion can fail:

$$\overline{\bigcup_{n=1}^{\infty} \{\tfrac{1}{n}\}} \neq \bigcup_{n=1}^{\infty} \overline{\{\tfrac{1}{n}\}}.$$

Let $x \in \overline{\bigcup \mathcal{A}}$. Let U be a nbhd of x that intersects only finitely many members of $\mathcal{A}$, say $A_1, \dots, A_n$.

If $x \notin \bar{A}_1 \cup \bar{A}_2 \cup \dots \cup \bar{A}_n$, then $\exists$ nbhd V of x that misses $\bar{A}_1 \cup \bar{A}_2 \cup \dots \cup \bar{A}_n$. Then $U \cap V$ is a nbhd of x that misses every $A \in \mathcal{A}$. Hence $U \cap V$ misses $\bigcup \mathcal{A}$ and so x is not in $\bigcup \mathcal{A}$ nor is x a limit pt of $\bigcup \mathcal{A}$. But then $x \notin \overline{\bigcup \mathcal{A}}$. Thus, $x \in \bar{A}_i$ for some $i \in \{1, \dots, n\}$. Therefore $x \in \bigcup \bar{\mathcal{A}}$. $\boxed{+}$

Lemma 39.2 Let X be metrizable. If $\mathcal{A}$ is an open covering of X, then $\exists$ a countable locally finite open covering $\mathcal{B}$ of X that refines $\mathcal{A}$.

Recall (§10) A set S with an order relation $<$ is said to be <u>well ordered</u> if every nonempty subset of S has a smallest element. The Well Ordering Thm said that every set can be well-ordered.

Pf of Lem. 9.2

Let $<$ be a well-ordering of $\mathcal{U}$.

$\forall\, U \in \mathcal{U}$ let $S_n(U) = \{x \in U \mid B(x, \frac{1}{n}) \subset U\}$. This is called a shrinking of U. It can be ϕ. Ex: $S_{10}([0,1)) = [\frac{1}{10}, \frac{9}{10}]$.

$\forall\, U \in \mathcal{U}$ let $T_n(U) = S_n(U) - \bigcup_{\substack{V \in \mathcal{U} \\ V < U}} V$.

$\forall\, U \in \mathcal{U}$ let $E_n(U) = \bigcup_{x \in T_n(U)} B(x, \frac{1}{3n})$. Each $E_n(U)$ is open.

The idea is we shrink each $U \in \mathcal{U}$, chop away $V \in \mathcal{U}$ that are $< U$ and the fatten up the result a bit to get $E_n(U)$.

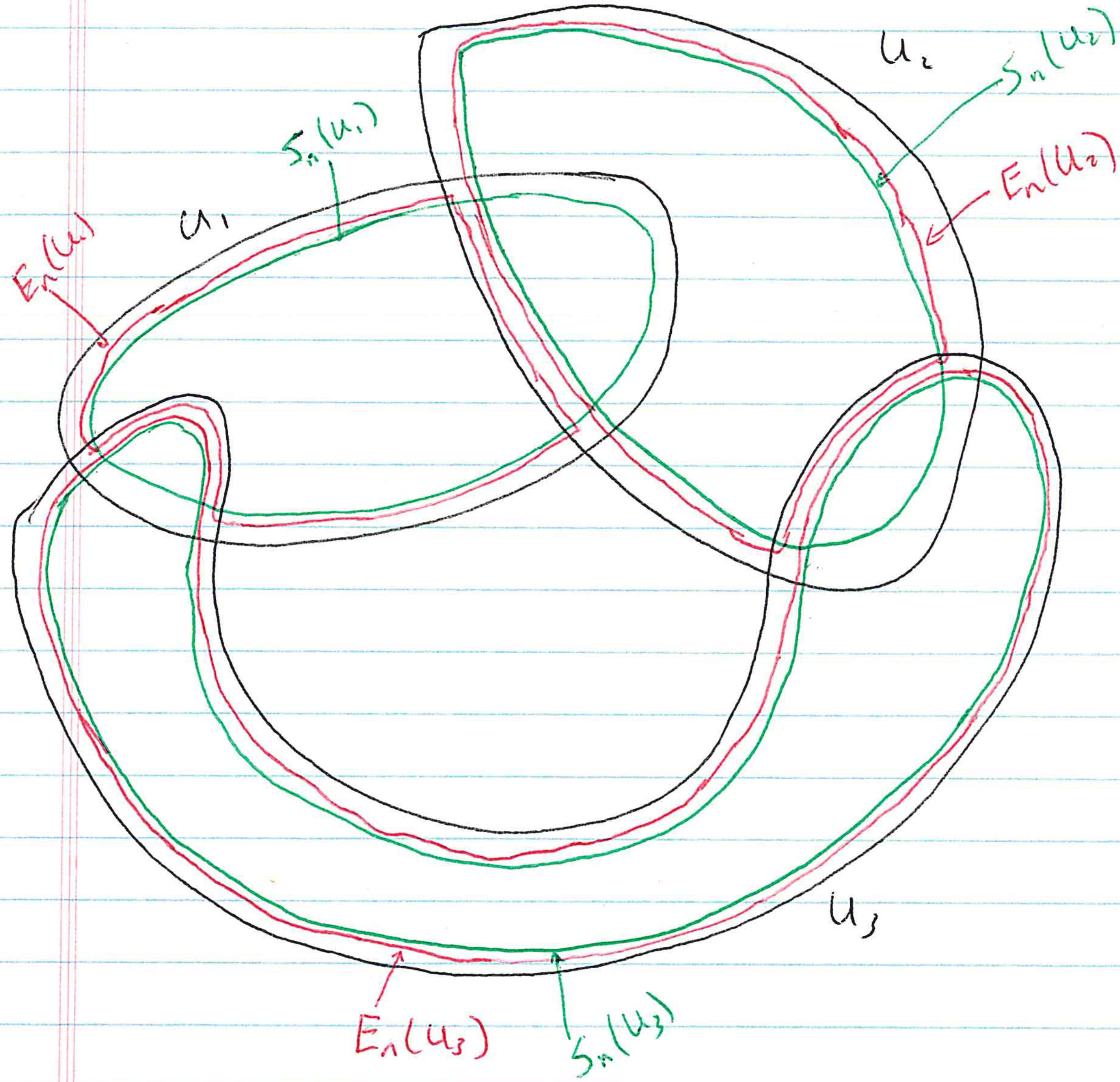
See the next page for an example.

If $V < U$, then $E_n(V)$ and $E_n(U)$ are disjoint. The proof is a triangle ineq exercise. Do tho. Thus $\{E_n(U) \mid U \in \mathcal{U}\}$ is a disjoint collection of open sets. Further,

$$\{E_n(U) \mid U \in \mathcal{U},\ n = 1, 2, 3 \ldots\}$$

is a countably loc. fin. open collection. It refines $\mathcal{U}$. We will show it covers X.

$U_1 > U_2 > U_3$



$E_n(U_1)$, $E_n(U_2)$ and $E_n(U_3)$ are disjoint.

Exercise: Redraw this for $U_3 > U_2 > U_1$.

Let $x \in X$. Let U be the least element of $\mathcal{U}$ that contains x. Let n be s.t. $B(x, \frac{1}{n}) \subset U$. Then $x \in S_n(U)$ by def. Since there are no $V < U$ that contain x, $x \in T_n(U) \subset E_n(\mathcal{U})$. $\boxtimes$