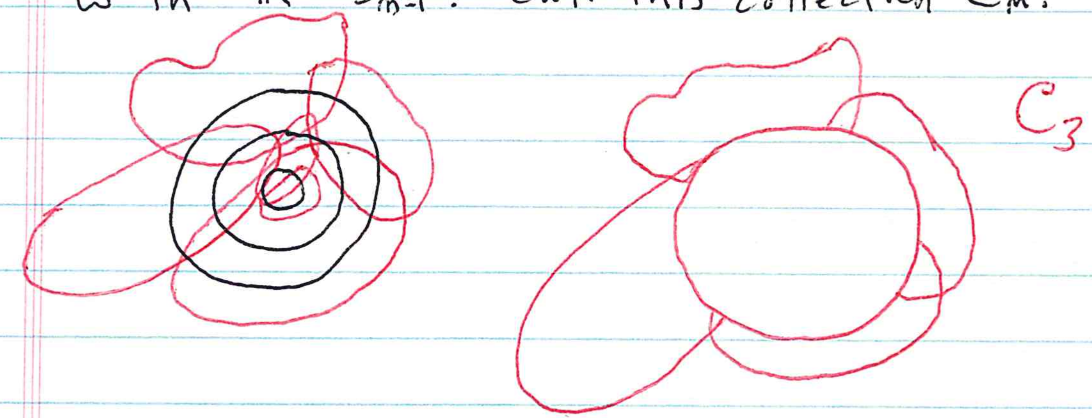
§41 **Paracompactness**

Def A top. sp. X is **paracompact** if every open covering of X has a locally finite open refinement that covers X.

Ex $\mathbb{R}^n$ is paracompact.

Pf Let $\mathcal{A}$ be an open cover of $\mathbb{R}^n$. Let $B_0 = \phi$. Let $B_m = B(0,m)$ for each $m \in \mathbb{N}$. For each $m \geq 1$ choose a finite subcollection of $\mathcal{A}$ that covers $\overline{B}_m$. Take each of these and form the intersection with $\mathbb{R}^n - \overline{B}_{m-1}$. Call this collection $\mathcal{C}_m$.



Let $\mathcal{C} = \bigcup_{m=1}^{\infty} \mathcal{C}_m$. We claim $\mathcal{C}$ is a loc. f. open refinement of $\mathcal{A}$ that covers $\mathbb{R}^n$.

$\mathcal{C}$ is a cover, since for $x \in \mathbb{R}^n$, $\exists m \geq$ s.t. $x \in B_m - \overline{B}_{m-1}$, and hence $\exists C \in \mathcal{C}_m$ s.t. $x \in C$.
$\mathcal{C}$ is loc. f. since for $x \in \mathbb{R}^n$ the nbhd B_m (for some m) only meets those sets in $\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_m$.
It is clear each $C \in \mathcal{C}$ is open since $C = A \cap (\mathbb{R}^n - \overline{B}_{m-1})$ for some m. This also shows $C \subset A \in \mathcal{A}$.
and $A \in \mathcal{A}$

Some basic facts.

Thm 41.2 Closed subspaces of paracompact spaces are paracompact

Ex 4 The product of even two paracompact spaces need not be paracompact. We will see shortly that $\mathbb{R}_\ell$ is paracompact, but that $\mathbb{R}_\ell^2$ is not.

Thm 41.1 Paracompact and Hausdorff implies normal.

Thm 41.4 Metrizable implies paracompact.

Thm 41.5 Regular and Lindelöf implies paracompact.

Proofs of 41.2 and 41.1 are easy, you can read them.

The proofs of 41.4 and 41.5 are easy once we have Lemma 41.3. We prove it below.

Ex 4: $\mathbb{R}_\ell$ is paracompact since it is Reg. and Lin, But $\mathbb{R}_\ell \times \mathbb{R}_\ell$ is not paracompact since it is Hausdorff but not normal.

Exercise #2 shows that the product of a paracompact space and a compact space is paracompact.

Lemma 41.3 Let X be a regular sp. Then (1), (2), (3) and (4) are equivalent, where:

(1) $\forall$ open covering of X, $\exists$ a refinement that is an open ct. loc. f. cover of X

(2) " " " " " " " " " a loc. f. cover of X.

(3) " " " " " " " " " a closed loc. f. cover of X.

(4) " " " " " " " " " an open loc. f. cover of X.

Note The real pt of this is (1) $\Rightarrow$ (4). (4) $\Rightarrow$ (1) is trivial and (2) and (3) are steps to get from (1) to (4).

(1) $\Rightarrow$ (2)

Pf Let $\mathcal{A}$ be an open covering of X and let $\mathcal{B}$ be an open ct. loc. f. refinement of $\mathcal{A}$ that covers X. Write $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$, where each $\mathcal{B}_n$ is loc. f.

Let $V_n = \bigcup_{U \in \mathcal{B}_n} U$. Let $S_n(U) = U - \bigcup_{i<n} V_i$ for $U \in \mathcal{B}_n$.

Let $\mathcal{C}_n = \{ S_n(U) \mid U \in \mathcal{B}_n \}$. This is a ref. of $\mathcal{B}_n$.

Let $\mathcal{C} = \bigcup_{n=1}^{\infty} \mathcal{C}_n$. We claim $\mathcal{C}$ covers X and is a loc. f. ref. of $\mathcal{A}$.

Let $x \in X$. For some n, $\exists\, U \in \mathcal{B}_n$ s.t. $x \in U$. Let N be the smallest such n. Since $x \notin V$ for any $V \in \mathcal{B}_i$ with $i < N$, then $x \in S_N(U)$. Thus $\mathcal{C}$ is a cover of X.

Since $\mathcal{B}_n$ is loc. f., for $n = 1, 2, \ldots, N$, $\exists$ a nbhd of x, W_n, that meets only finitely many elements of $\mathcal{B}_n$. Since $S_N(U) \subset U$, W_n only meets finitely many elements of $\mathcal{C}_n$, $n = 1, 2, \ldots, N$.

Since $U \in \mathcal{B}_N$, U meets no members of $\mathcal{C}_i$ for $i < N$. Thus,

$$W_1 \cap W_2 \cap \cdots \cap W_N \cap U$$

is a nbhd of x that meets only finitely many members of $\mathcal{C}$. Thus, $\mathcal{C}$ is loc. f.

(2) $\Rightarrow$ (3) Let $\mathcal{A}$ be an open covering of X. Let $\mathcal{B}$ be the collection of all open sets $U \subset X$ s.t. $\bar{U} \subset A$ for some $A \in \mathcal{A}$. Then $\mathcal{B}$ is an open cover of X by Lemma 31.1 a. [X reg $\Rightarrow$ $\forall x \in A$, $\exists U$ open s.t. $x \in U$, $\bar{U} \subset A$.] $\mathcal{B}$ is a ref. of $\mathcal{A}$.

By assumption (2) $\exists$ a loc. f. ref that covers X. Call it $\mathcal{C}$. Let $\mathcal{D} = \{\bar{C} \mid C \in \mathcal{C}\}$. Clearly $\mathcal{D}$ covers X. It is loc. f. by Lemma 39.1.

For each $C \in \mathcal{C}$, $\exists$ $B \in \mathcal{B}$ and $A \in \mathcal{A}$ s.t. $C \subset B$, $\bar{B} \subset A$. Thus $\bar{C} \subset \bar{B} \subset A$. So, $\mathcal{D}$ is a ref. of $\mathcal{A}$, that is closed and is loc. f. covering of X.

(3) ⇒ (4) Let $\mathcal{A}$ be an open cover of X and let $\mathcal{B}$ be a closed loc. f. ref. of $\mathcal{A}$ that covers X. We will "expand" the closed members of $\mathcal{B}$ to open sets that will form a loc. f. ref $\mathcal{A}$. Obviously it covers X.

For each $x \in X$, $\exists$ a nbhd U of x that meets only finitely many members of $\mathcal{B}$. Thus, the collection $\mathcal{U}$ of all such open sets (that meets only finitely many members of $\mathcal{B}$) is an open covering of X.

By assumption (3) $\exists$ a closed ref. $\mathcal{C}$ of $\mathcal{U}$ that covers X and is loc. f. Since $\mathcal{C}$ ref's $\mathcal{U}$ every member of $\mathcal{C}$ meets only finitely many members of $\mathcal{B}$.

$\forall B \in \mathcal{B}$, define $\mathcal{C}(B) = \{C \in \mathcal{C} \mid C \subset X - B\}$, and

define $E(B) = X - \bigcup_{C \in \mathcal{C}(B)} C$. (E for Expansion)

Clearly $B \subset E(B)$ since all $C \in \mathcal{C}(B)$ miss B. We claim $E(B)$ is open. By Lemma 39.1 (a) each $\mathcal{C}(B)$ is loc. f. By Lemma 39.1 (c), $\bigcup \bar{C} = \overline{\bigcup C}$ (over $C \in \mathcal{C}(B)$), but $\bigcup C = \bigcup \bar{C}$ so $\bigcup C$ is closed. Thus, $E(B)$ is open.

Unfortunately $\{E(B)\}_{B\in\mathcal{B}}$ may not be a ref. of $\mathcal{A}$. We trim $E(B)$ down a bit as follows. Let $F(B)$ be a member of $\mathcal{A}$ that contains B. Then define

$$\mathcal{D} = \{ E(B) \cap F(B) \mid B \in \mathcal{B} \}.$$

Clearly, $\mathcal{D}$ is a ref. of $\mathcal{A}$. Its members are all open since $E(B)$ and $F(B) \in \mathcal{A}$ are open. Since $B \subset F(B) \cap F(B)$ and $\mathcal{B}$ covers X, then $\mathcal{D}$ covers X.

We claim $\mathcal{D}$ is loc. f. Let $x \in X$. Recall $\mathcal{C}$ was loc. f. Let W be a nbhd of x that meets only finitely many members of $\mathcal{C}$, say $C_1, C_2, \ldots, C_k \in \mathcal{C}$. Since $\mathcal{C}$ covers X, $W \subset C_1 \cup C_2 \cup \cdots \cup C_k$.

We claim each member of $\mathcal{C}$ meets only finitely many members of $\mathcal{D}$. This will imply W meets only finitely many members of $\mathcal{D}$ and we will be done.

Let $C \in \mathcal{C}$. Then C meets only finitely many members of $\mathcal{B}$. Suppose C meets $E(B) \cap F(B) \in \mathcal{D}$. We claim $C \cap B \neq \emptyset$. Suppose $C \cap B = \emptyset$. We know $C \cap E(B) \neq \emptyset$. But $C \subset X - B$. But in defining $E(B)$ we cut out all such members of $\mathcal{C}$. Thus $C \not\subset X - B$, so $C \cap B \neq \emptyset$.

Partitions of Unity

What's all this good for? In §36 we established the existence of partitions of unity wrt finite open coverings of a normal space. Below we generalized the def. of a part. of unity and show they exist of arbitrary open covers of paracompact Hausdorff spaces.

Def Let $\{U_\alpha\}$ be an open covering of X. A family of cont. func's

$$\phi_\alpha : X \to [0,1]$$

is a partition of unity wrt $\{U_\alpha\}$ if

① $\operatorname{supp}(\phi_\alpha) \subset U_\alpha$
② $\{\operatorname{supp}(\phi_\alpha)\}$ is loc. finite
③ $\sum \phi_\alpha(x) = 1 \quad \forall x \in X$.

Note that the sum in ③ is well defined by ②, where we take the sum to be of nonzero terms.

Thm (41.7) Let X be a paracompact Hausdorff sp. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in J}$ be an open covering of X. Then $\exists$ partition of unity $\{\phi_\alpha : X \to [0,1]\}$ wrt $\mathcal{U}$.

The proof uses the "shrinking lemma," Lemma 41.6

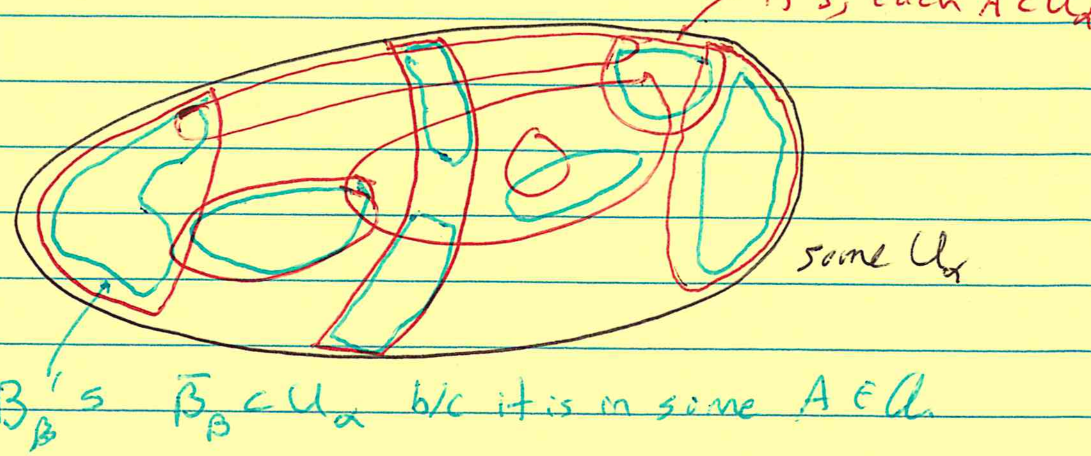
Lemma 41.6 Let X be a paracpt Haus. sp. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in J}$ be an open covering of X. $\exists\ \mathcal{V} = \{V_\alpha\}_{\alpha \in J}$ that is an open loc. f. covering of X s.t. $\overline{V}_\alpha \subset U_\alpha,\ \forall \alpha \in J$.

Proof Let $\mathcal{A} = \{A \mid A \text{ is open, } \overline{A} \subset U_\alpha \text{ for some } \alpha \in J\}$. Since X is reg. $\mathcal{A}$ covers X. [$\forall x \in X, \alpha \in J$ s.t. $x \in U_\alpha$, $\exists$ open set W s.t. $x \in W, \overline{W} \subset U_\alpha$.]

Let $\mathcal{B}$ be an open loc. f. ref. of $\mathcal{A}$ that covers X. [$\mathcal{B}$ exists b/c X is paracpt.] Index the members of $\mathcal{B}$ with some index set K: $\mathcal{B} = \{B_\beta\}_{\beta \in K}$.

Define $f: K \to J$ by, for each $\beta \in K$, choose an $\alpha \in J$ s.t. $\overline{B}_\beta \subset U_\alpha$, let $f(\beta) = \alpha$. Check that $f(\beta)$ is always defined for $\beta \in K$.

For each $\alpha \in J$, let $V_\alpha = \bigcup_{f(\beta)=\alpha} B_\beta$ and $\mathcal{B}_\alpha = \{B_\beta \in \mathcal{B} \mid f(\beta) = \alpha\}$. (These can be empty.)



Since each $B_\beta \in \mathcal{B}_\alpha$ is a subset of a member $A \in \mathcal{A}$ with $\bar{A} \subset \mathcal{U}_\alpha$, we have $\overline{B}_\beta \subset \mathcal{U}_\alpha$ whenever $\alpha = f(\beta)$.

Since each $\mathcal{B}_\alpha$ is loc. f., we have by Lemma 39.1(c)

$$\overline{V}_\alpha = \overline{\bigcup_{f(\beta)=\alpha} B_\beta} = \bigcup_{f(\beta)=\alpha} \overline{B_\beta} \subset \mathcal{U}_\alpha .$$

It is only left the check that $\{V_\alpha\}_{\alpha \in J}$ is loc. f.

Let $x \in X$. $\exists$ nbhd W of x that meets only finitely many members $B_\beta \in \mathcal{B}$ [$\mathcal{B}$ is loc. f.]. Suppose their indexes are $\beta_1, \beta_2, \dots, \beta_k$. Then W meets V_α only if $\alpha \in \{f(\beta_1), \dots, f(\beta_k)\}$.

$\square$

Proof of Thm 41.7: Paracompact + Hausdorff $\Rightarrow \exists$ part. unit

Apply the shrinking lemma twice to produce two loc. f. collections of open sets $\{W_\alpha\}$ and $\{V_\alpha\}$ that cover X s.t. $\bar{W}_\alpha \subset V_\alpha$, $\bar{V}_\alpha \subset U_\alpha$ $\forall \alpha \in J$.

Since X is normal, by Urysohn's Lemma, $\forall \alpha \in J$ $\exists\, \phi_\alpha : X \to [0,1]$ s.t. $\phi_\alpha(\bar{W}_\alpha) = \{1\}$ and $\phi_\alpha(X - V_\alpha) = \{0\}$. It follows that $\operatorname{supp}(\phi_\alpha) \subset \bar{V}_\alpha \subset U_\alpha$.

By Lemma 39.1b, $\{\bar{V}_\alpha\}$ is loc. f. Hence the collection $\{\operatorname{supp} \phi_\alpha\}$ is loc. f.

Now consider $\Phi(x) = \sum_{\alpha \in J} \phi_\alpha(x)$. By loc. finiteness the sum can be interperted as a finite sum. Since $\{W_\alpha\}$ covers X, the sum is never 0.

We claim Φ is cont. Let $x \in X$ and W be a nbhd of x that meets only finitey many member of $\{\operatorname{supp}(\phi_\alpha)\}$. Since $\Phi(x)$ is a finite some of cont. func's on W it is cont. on W and hence a x. Since this holds for any $x \in X$, Φ is cont.

Finally define $\phi_\alpha'(x) = \dfrac{\phi_\alpha(x)}{\Phi(x)}$. Then $\{\phi_\alpha'\}$ is the desired partition of unity wrt $\{U_\alpha\}$. $\square$