§48 Baire Spaces

Note $A^\circ = \emptyset \Leftrightarrow \overline{X - A} = X$, ie. empty interior $\Leftrightarrow$ dense complement

Def A complete metric sp. is a met. sp. in which all Cauchy sequence converge. (see §43.)

Def A top. sp X is a Baire space if whenever $\{A_n\}$ is a countable collection of closed sets with empty interiors we have that $\cup A_n$ has empty interior.

$$\left(\overline{A_n} = A_n,\ A_n^\circ = \emptyset,\ n = 1,3,3 \ldots\right) \Rightarrow \left(\bigcup_{n=1}^{\infty} A\right)^\circ = \emptyset.$$

This can be reformulated as (Lemma 48.1):

if $\{U_n\}$ is a ct. coll. of open dense subsets then $\bigcap_{n=1}^{\infty} U_n$ is dense, ie.

$$\left(U_n^\circ = U_n,\ \overline{U_n} = X,\ n = 1,2,3 \ldots\right) \Rightarrow \overline{\bigcap_{n=1}^{\infty} U_n} = X.$$

Ex Here is an example of an open dense set. $\{(a,b,c,d) \in \mathbb{R}^4 \mid \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertable$\}$. Why?

Lemma 48.3 Let $C_1 \supset C_2 \supset C_3 \supset \cdots$ be a nested seq. of nonempty closed sets is a complete metric space X. If $\operatorname{diam} C_n \to 0$, then $\bigcap C_n \neq \emptyset$.

Pf See textbook.

We had this before for compact sets, (see pg 170) We did not even ~~not~~ need X to be metric.

Note: Not true for open: $\bigcap_{n=1}^{\infty} (0, \frac{1}{n}) = \emptyset$.

Not true if $\operatorname{diam} \not\to 0$,

$$\bigcap_{n=1}^{\infty} [n, \infty) = \emptyset.$$

Thm 48.2 ①* If X is a complete metric sp., then X is a Baire sp.
② If X is compact and Hausdorff, then X is a B. sp.

Pf ① Let X be a complete met. sp. Thus X is reg. Let $\{A_n\}$ be closed, countable, $\operatorname{int} A_n = \emptyset$. Let $A = \bigcup A_n$. We will show $\operatorname{int} A = \emptyset$.

Suppose $U_0 \neq \emptyset$ is open in X. We will prove that $U_0 \not\subset A$. Since U_0 was abitrary, $\operatorname{int} A = \emptyset$.

Since $\operatorname{int} A_1 = \emptyset$, we know $U_0 \not\subset A_1$. Choose $y \in U_0 - A_1$. By reg. $\exists\, U_1$ open, $y \in U_1$, $\overline{U}_1 \subset U_0 - A_1$ (which is open). Thus, $\overline{U}_1 \cap A_1 = \emptyset$. We can replace U_1 with any open ball $B(y, r)$, with $r < 1$. Wlog, we assume $\operatorname{diam} U_1 < 1$.

* We did this in 452

Now we repeat this. U_1 is a nonempty open set. Since $\operatorname{int} A_2 = \emptyset$, $U_1 \not\subset A_2$. Pick $y \in U_1 - A_2$. By reg. $\exists$ open U_2 s.t.

$$y \in U_2, \ \bar{U}_2 \subset U_1 - A_2.$$

Thus $\bar{U}_2 \cap A_2 = \emptyset$. Wlog assume diam $U_2 < \frac{1}{2}$.

Repeat. $U_2 \not\subset A_3$. Pick $y \in U_2 - A_3$. $\exists$ U_3 open s.t.

$$y \in U_3, \ \bar{U}_3 \subset U_2 - A_2 \quad (\Rightarrow \bar{U}_3 \cap A_2 = \emptyset)$$

and we assume diam $U_3 < \frac{1}{3}$. Etc etc.

We generate a seq $\{U_n\}_{n=1}^{\infty}$ of open nonempty sets s.t.

$$\bar{U}_n \subset U_{n-1}, \quad \text{diam } U_n < \frac{1}{n},$$

$$\bar{U}_n \cap A_n = \emptyset.$$

By Lemma 48.3 $\cap \bar{U}_n \neq \emptyset$. Let $x \in \cap \bar{U}_n$. Then $x \in U_0$ ($x \in \bar{U}_1 \subset U_0$) and $x \notin A_n$, $n > 1$. Thus, $x \notin A = \cup A_n$. Hence $U_0 \not\subset A$. $\square$

Pf ② Let X be compact and Hausdorff. Then X is normal and hence reg. The proof is similar except diam U_n makes no sence. But we can still construct $\{U_n\}$ with

$$\overline{U}_1 \supset \overline{U}_2 \supset \overline{U}_3 \supset \overline{U}_4 \supset \overline{U}_5 \supset \cdots$$

Now $\{\overline{U}_n\}$ has the F.I.P. Since X is compact,

$$\cap \overline{U}_n \neq \phi. \qquad \square$$

Exercise #3 Loc comp. and Hausdorff $\Rightarrow$ Baire.

Outline of Pf By #3 in §32 Loc comp and H. $\Rightarrow$ Reg. To get $\cap \overline{U}_n \neq \phi$, work in the unique one pt compactification of the space. You have to check that $\infty \notin \overline{U}_n$, $n = 1, 2, 3, \ldots$. But this is easy by just choosing U_0 s.t. $\infty \notin U_0$.

Exercise #11 Show that $\mathbb{R}_\ell$ is a Baire space.