§51 # Homotopy of Paths

Def If f and g are continuous maps from a top. sp. X into a top. sp. Y, then we say f is homotopic to g if $\exists$ a cont. map $H : X \times I \to Y$ s.t. $(I = [0,1])$

$$H(x,0) = f(x) \quad \forall x \in X$$
$$H(x,1) = g(x) \quad \forall x \in X.$$

The map H is called a homotopy. If f is homotopic to a constant function we say f is nulhomotopic.

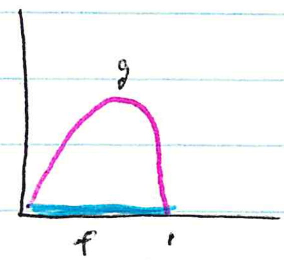
Def Let $A \subset X$. If f and g are cont. maps from X to Y that agree on A, we say they are homotopic relative to A if $\exists$ a cont. map $H : X \times I \to Y$ s.t.

$$H(x,0) = f(x) \quad \forall x \in X$$
$$H(x,1) = g(x), \quad \forall x \in X$$
$$H(a,t) = f(a) = g(a), \quad \forall a \in A, t \in I.$$

Let f and g be two paths in Y with the same end pts. They are path homotopic if they are homotopic rel. $\{0,1\}$, that is the homotopy holds the end pts fixed. Thus $\exists$ a cont. $H : I \times I \to Y$ s.t.

$$H(s,0) = f(s) \quad \forall s \in I$$
$$H(s,1) = g(s) \quad \forall s \in I$$
$$H(0,t) = f(0) = g(0) \quad \forall t \in I$$
$$H(1,t) = f(1) = g(1) \quad \forall t \in I.$$

Ex Let $f: I \to \mathbb{R}^2$ be $f(s) = (s, 0)$.
Let $g: I \to \mathbb{R}^2$ be $g(s) = (s, \sin(\pi s))$



Let $H(s,t) = (s, t\sin(\pi s))$. Then

$$H(s,0) = (s,0) = f(s) \quad \forall s \in I$$
$$H(s,1) = (s, \sin(\pi s)) = g(s) \quad \forall s \in I$$
$$H(0,t) = (0,0) \quad \forall t \in I$$
$$H(1,t) = (1,0) \quad \forall t \in I.$$

Thus, $f \simeq_p g$.

Now let $X = \mathbb{R}^2 - \{(\frac{1}{2}, \frac{1}{2})\}$. Then H is no longer a valid homotopy since $H(\frac{1}{2}, \frac{1}{2}) = (\frac{1}{2}, \frac{1}{2}) \notin X$. But, how can we prove that there is no path homotopy from f to g in X?

Lemma 5.1 Homotopy and relative homotopy are equivalence relations.

Pf Let f, g, h be cont. maps from X to Y that agree on $A \subset X$.

$f \simeq_A f$ is obvious; just use $H(x,t) = f(x)$.

Suppose $f \simeq_A g$ and let H be a homotopy rel. A from f to g. Let $\bar{H}(x,t) = H(x, 1-t)$. This will give a homotopy rel A from g to f. Check this! Thus $g \simeq_A f$.

Suppose $f \simeq_A g$ and $g \simeq_A h$. Let F be a homotopy rel A from f to g and G be a homotopy rel. A from g to h. Define $H: X \times I \to Y$ by

$$H(x,t) = \begin{cases} F(x, 2t) & t \in [0, \frac{1}{2}] \\ G(x, 2t-1) & t \in [\frac{1}{2}, 1]. \end{cases}$$

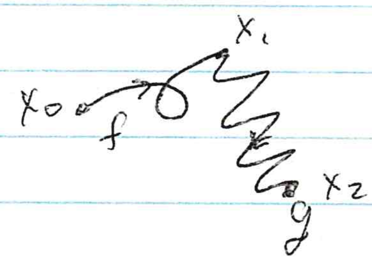
H is cont. by the pasting lemma (pg 108).

$$H(x,0) = F(x,0) = f(x) \quad \forall x \in X$$
$$H(x,1) = G(x,1) = g(x) \quad \forall x \in X$$
$$H(a,t) = F(a,2t) = f(a) = g(a) = h(a) \quad \forall t \in [0, \tfrac{1}{2}], \ a \in A$$
$$H(a,t) = G(a, 2t-1) = g(a) = h(a) = f(a) \quad \forall t \in [\tfrac{1}{2}, 1]$$
$$\Rightarrow H(a,t) = h(a) = f(a) \quad \forall a \in A, \ t \in I.$$

Def Let f be a path in X from x_0 to x_1 and g be a path in X from x_1 to x_2. Define $f*g$ to be the path given by

$$(f*g)(s) = \begin{cases} f(2s) & s \in [0,\frac{1}{2}] \\ g(2s-1) & s \in [\frac{1}{2},1]. \end{cases}$$

Then $f*g$ is a path in X from x_0 to x_2.

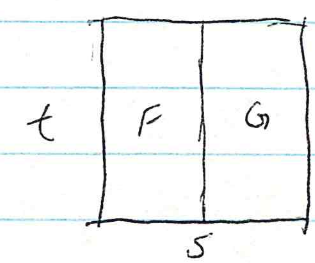


Def For a path f, let $[f]$ denotes the path eq. class of f.

Lemma $[f*g] = [f'*g]$ $\forall$ $f' \in [f]$, $g' \in [g]$.
(Thus we can define $[f]*[g] = [f*g]$ unambiguously.)

Pf Let F be a homotopy from f to f' and G a homotopy from g to g'. Let $H: I \times I \to Y$ be

$$H(s,t) = \begin{cases} F(2s,t) & s \in [0,\frac{1}{2}], t \in [0,1] \\ G(2s-1,t) & s \in [\frac{1}{2},1], t \in [0,1]. \end{cases}$$



$$f'(1)$$

For $s=\frac{1}{2}$, $F(1,t) = f(1) \;\forall t\in I$, and

$$G(0,t) = g(0) = g'(0) \quad \forall t\in I.$$

By assumption $f(1)=f'(1)=g(0)=g'(0)$. Thus H is cont. by the pasting lemma. You can check that

$$H(s,0) = (f*g)(s)$$
$$H(s,1) = (f'*g')(s)$$
$$H(0,t) = f(0)=f'(0)$$
$$H(1,t) = g(1)=g'(1).$$

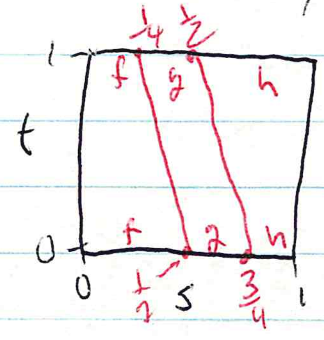
$\blacksquare$

Notation Let $x\in X$. Let $e_x : I\to X$ be $e_x(s)=x$.
Given $f: I\to X$, define $\bar{f}(s) = f(1-s)$.
Below, let $x_0 = f(0)$ and $x_1 = f(1)$.

Thm (51.2) ① $*$ is assoc., $[f]*([g]*[h]) = ([f]*[g])*h$.
② $[f]*[e_{x_1}] = [f]$ and $[e_{x_0}]*[f]=[f]$.
③ $[f]*[\bar{f}] = [e_{x_0}]$ and $[\bar{f}]*[f] = [e_{x_1}]$.

Pf ① It is enough to show $f*(g*h) \simeq_p (f*g)*h$.

To construct our homotopy we consider the schematic



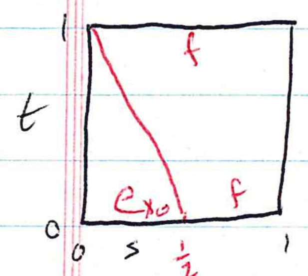
Let $H: I \times I \to X$ be given by

$$H(s,t) = \begin{cases} f\left(\frac{s}{\frac{1}{2} - \frac{1}{4}t}\right) & s \in [0, \frac{1}{2} - \frac{1}{4}t] \\ g\left(\frac{s - (\frac{1}{2} - \frac{1}{4}t)}{\frac{1}{4}}\right) & s \in [\frac{1}{2} - \frac{1}{4}t, \frac{3}{4} - \frac{1}{4}t] \\ h\left(\frac{s - (\frac{3}{4} - \frac{1}{4}t)}{1 - (\frac{3}{4} - \frac{1}{4}t)}\right) & s \in [\frac{3}{4} - \frac{1}{4}t, 1]. \end{cases}$$

To show H is cont. substitute $s = \frac{1}{2} - \frac{1}{4}t$ into the first two formulas and $s = \frac{3}{4} - \frac{1}{4}t$ into the latter two formulas. To check that H is a path homotopy taking $f*(g*h)$ to $(f*g)*h$ show

$$H(s,0) = (f*(g*h))(s)$$
$$H(s,1) = ((f*g)*h)(s)$$
$$H(0,t) = f(0)$$
$$H(1,t) = h(1).$$

(2) We only need to show that $e_{x_0} * f \simeq_p f$.



Let $H(s,t) = \begin{cases} x_0 & s \in [0, \frac{1}{2} - \frac{1}{2}t] \\ f\left(\frac{s - (\frac{1}{2} - \frac{1}{2}t)}{1 - (\frac{1}{2} - \frac{1}{2}t)}\right) & s \in [\frac{1}{2} - \frac{1}{2}t, 1]. \end{cases}$

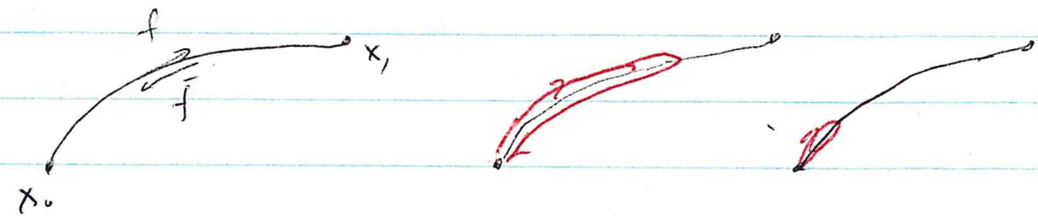
You can check continuity and

$$H(s,0) = (e_{x_0} * f)(s)$$
$$H(s,1) = f(s)$$
$$H(0,t) = x_0$$
$$H(1,t) = f(1).$$

The proof that $[f] * [e_{x_1}] = [f]$ is similar. Try working it out.

③ We need to show $f * \bar{f} \simeq e_{x_0}$ to get $[f] * [\bar{f}] = [e_{x_0}]$. The proof that $[\bar{f}] * [f] = [e_{x_1}]$ is similar.

Idea:



Note $$(f * \bar{f})(s) = \begin{cases} f(2s) & 0 \le s \le \frac{1}{2} \\ \bar{f}(2s-1) & \frac{1}{2} \le s \le 1 \end{cases} = \begin{cases} f(2s) & 0 \le s \le \frac{1}{2} \\ f(1-(2s-1)) = f(2(1-s)) & \frac{1}{2} \le s \le 1 \end{cases}$$

Let $$H(s,t) = \begin{cases} f(2s(1-t)) & s \in [0, \frac{1}{2}] \\ f(2(1-s)(1-t)) & s \in [\frac{1}{2}, 1]. \end{cases}$$

It is easy to check H is cont. and that

$$\begin{aligned} H(s,0) &= (f * \bar{f})(s) \\ H(s,1) &= e_{x_0}(s) = x_0 \\ H(0,t) &= f(0) = x_0 \\ H(1,t) &= f(0) = x_0 \end{aligned}$$

Compute $H(s, \frac{3}{4})$, $H(s, \frac{1}{2})$, $H(s, \frac{1}{4})$ and you'll see what it is doing.

$\square$