Section 52

Def Let (X, x_0) be a pointed space, i.e. a top sp X together with a base point $x_0 \in X$.

Let $\Omega(X, x_0) = \{ \text{paths } f: I \to X \text{ s.t. } f(0) = f(1) = x_0 \}$. Such paths are called loops based at x_0.

Let $\pi_1(X, x_0) = \Omega / \simeq_p = \{ [f] \mid f: I \to X \text{ is a path based at } x_0 \}$. Then using $*$ this is a group called the fundamental group of X based at x_0; it is also called the first homotopy group.

Ex $\pi_1(\mathbb{R}^n, 0)$ is the trivial group. You can prove this by showing any loop based at the origin can be homotoped to e_0.

π_1 of any ball with any base point is trivial.

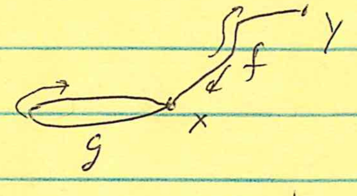
Def A space X is simply conn'd if it is path conn'd and every loop is nulhomotopic.

Fact It is easy to show X is simply conn'd iff $\pi_1(X, x_0) =$ trivial. (any x_0 will do).

Thm Let X be path conn'd and let $x, y \in X$. Then there is an isomorphism between $\pi_1(X,x)$ and $\pi_1(X,y)$. [But they are not equal

Pf Let f be a path from x to y. Define $\hat{f}: \pi_1(X,x) \to \pi_1(X,y)$ by

$$\hat{f}([g]) = [\bar{f}]*[g]*[f].$$



① $\hat{f}$ is well defined. $\bar{f}$ ends at x and g starts at x, so $\bar{f}*g$ is defined. g ends at x and f starts at x so, $\bar{f}*g*f$ is defined. $\bar{f}$ starts at y and f ends at y so $\bar{f}*g*f$ is a loop based at y. Thus $[\bar{f}*g*f] \in \pi_1(X,y)$

② $\hat{f}$ is a homomorphism. Let $[g], [h] \in \pi_1(X,x)$. Then

$$\begin{aligned}\hat{f}([g]*[h]) &= [\bar{f}]*[g]*[h]*[f] \\ &= [\bar{f}]*[g]*[f]*[\bar{f}]*[h]*[f] \\ &= \hat{f}([g]) * \hat{f}([h]).\end{aligned}$$

(Note the repeated use of associativity.)
(It is automatic that $\hat{f}([g]^{-1}) = (\hat{f}([g]))^{-1}$.)

③ $\hat{f}$ is one-to-one. We will show the kernel is trivial. Let $\hat{f}([g]) = [e_y]$. Then

$$\begin{aligned}[\bar{f}]*[g]*[f] &= [e_y] \\ [f]*[\bar{f}]*[g]*[f]*[\bar{f}] &= [f]*[e_y]*[\bar{f}] \\ [e_x]*[g]*[e_x] &= [f*e_y*\bar{f}] \\ [g] &= [e_x].\end{aligned}$$

(3) $\hat{f}$ is onto. Let $[h] \in \pi_1(X, y)$. Let $g = f * h * \bar{f}$.
Then $[g] \in \pi_1(X, x)$, and

$$\hat{f}([g]) = [\bar{f}] * [f] * [h] * [\bar{f}] * [f] = [h].$$

Thus $\hat{f}$ is an isomorphism.

Def Let $h : (X, x_0) \to (Y, y_0)$ be cont, $h(x_0) = y_0$.
Define $h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ by

$$h_*([f]) = [h \circ f].$$

We say h_* is induced by h.

Thm h_* is a well defined group homomorphism.

Pf Let f and g be path homotopic loops in X based at x_0.
For h_* to be "well defined" we need $h_*([f]) = h_*([g])$.
Thus we need for $h \circ f$ and $h \circ g$ to be path homotopic.
Let $F(s,t)$ be a path homotopy taking f to g. Thus

$$F(s,0) = f(s), \quad F(s,1) = g(s) \quad \forall s \in [0,1].$$
$$F(0,t) = F(1,t) = x_0 \quad \forall t \in [0,1].$$

Let $F' = h \circ F$. We claim F' is a path homotopy from $h \circ f$ to $h \circ g$ in Y.
You can check this.

Now we show that h_* is a homomorphism. Let $[f], [g] \in \pi_1(X,$
Then
$$h_*([f]*[g]) = h_*([f*g]) = [h(f*g)].$$
But,
$$(h(f*g))(s) = \begin{cases} h(f(2s)) & s \in [0, 1/2] \\ h(g(2s-1)) & s \in [1/2, 1], \end{cases}$$
which equals $(h\circ f * h\circ g)(s)$. Thus,
$$[h(f*g)] = [h\circ f * h\circ g] = [h\circ f] * [h\circ g]$$
$$= h_*([f]) * h_*([g]). \qquad \boxtimes$$

<u>Thm</u> (a) If $h: (X, x_0) \to (Y, y_0)$ is a homeo., then h_* is an isomorphism.

(b) If $h: (X, x_0) \to (Y, y_0)$ and $k: (Y, y_0) \to (Z, z_0)$ are cont. then $(k\circ h)_* = k_* \circ h_*$.

<u>Pf</u> Easy.

These are called "functorial" properties.

$$\begin{tikzcd} (X, x_0) \arrow[d, "h"] \arrow[r, phantom, "\xrightarrow[\text{functor}]{\pi_1}"] & \pi_1(X, x_0) \arrow[d, "h_*"] \\ (Y, y_0) & \pi_1(Y, y_0) \end{tikzcd}$$

We now know that π_1 is a top. invariant of path connected spaces.

#7 Let $(G,+)$ be a top. group with identity element 0. We will show that $\pi_1(G,0)$ is abelian.

(a) Let $\Omega(G,0)$ be the set of loops in G based at 0. We will show that $(\Omega(G,0),+)$ is a group, where $+$ means

$$(f+g)(s) = f(s)+g(s), \quad \forall s \in [0,1].$$

Pf

1. Closure: Since f and g are loops based at 0 $(f+g)(0) = 0+0=0$ and $(f+g)(1) = 0+0=0$. Thus $f+g : I \to G$ is a loop.

2. Associativity is immediate.

3. $e_0 : I \to \{0\}$ is the identity element.

4. Given f define $-f$ as $(-f)(s) = -f(s)$.

Thus, $\Omega(G,0)$ is a group under $+$.

(b) Show that $+$ respects path homotopy and thus induces a group operation on $\pi_1(G,0)$.

Pf Suppose $f \simeq_p f'$ and $g \simeq_p g'$. We claim $f+g \simeq_p f'+g'$.

Let $F(s,t)$ and $G(s,t)$ be path homotopies taking f to f' and g to g', resp. Let $H = F+G$. Then

$$H(s,0) = F(s,0)+G(s,0) = f(s)+g(s) = (f+g)(s) \quad \forall s \in I,$$
$$H(s,1) = F(s,1)+G(s,1) = f'(s)+g'(s) = (f'+g')(s) \quad \forall s \in I,$$
$$H(0,t) = F(0,t)+G(0,t) = f(0)+g(0) = 0+0=0 \quad \forall t \in I,$$
$$H(1,t) = F(1,t)+G(1,t) = f(1)+g(1) = 0+0=0 \quad \forall t \in I.$$

Thus, $f+g \simeq_p f'+g'$ and so $[f]+[g]=[f+g]$ is well defined. Therefore $(\pi_1(G,0),+)$ is a group.

(c) Show that $[f]*[g]=[f]+[g]$!!

Pf $[f]*[g] = [f*g] \overset{?}{=} [f*e_0 + e_0*g] = [f+g] = [f]+[g]$.

(?) $f*e_0 + e_0*g = \begin{Bmatrix} f(2s) \\ e_0(2s-1) \end{Bmatrix} + \begin{Bmatrix} e_0(2s) \\ g(2s-1) \end{Bmatrix} =$

$\begin{Bmatrix} f(2s)+0 \\ 0+g(2s-1) \end{Bmatrix} = \begin{Bmatrix} f(2s) \\ g(2s-1) \end{Bmatrix}$, where the upper term in each pair of braces is for $s \in [0,\frac{1}{2}]$ and the lower term is for $s \in [\frac{1}{2},1]$. But $f*g =$
$\begin{cases} f(2s) & s \in [0,\frac{1}{2}], \\ g(2s-1) & s \in [\frac{1}{2},1]. \end{cases}$

1 Show that $\pi_1(G,0)$, under $*$, is abelian.

Pf Although $+$ need not be abelian, we do know $0+x=x+0$.

$$[f]*[g]=[f*g]\overset{\checkmark}{=}[f*e_0+e_0*g]\overset{?}{=}[e_0*g+f*e_0]$$

$$\overset{\checkmark}{=}[g*e_0+e_0*f]\overset{\checkmark}{=}[g*f]=[g]*[f]$$

$\checkmark$ is what we showed in c.

$\checkmark$ since $g*e_0\simeq_p e_0*g$ and $f*e_0\simeq e_0*f$.

(?) First assume $s\in[0,\frac{1}{2}]$. Then

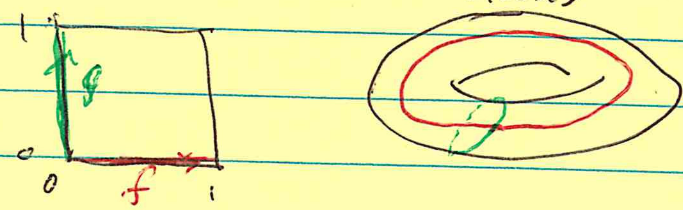
$$(f*e_0+e_0*g)(s)=f(s)+e_0(s)=e_0(s)+f(s)=(e_0*g+f*e_0)(s).$$

For $s\in[\frac{1}{2},1]$ we have

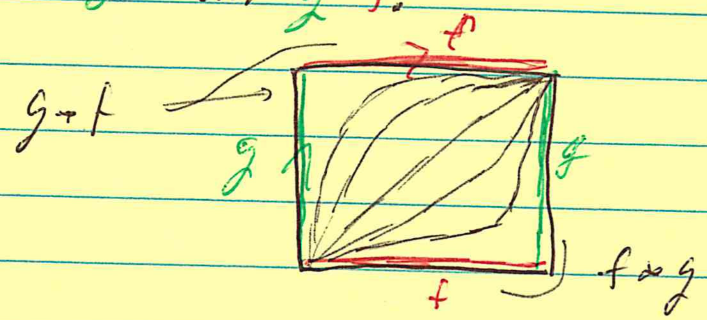
$$(f*e_0+e_0*g)(s)=e_0(s)+g(s)=g(s)+e_0(s)=(e_0*g+f*e_0)(s).$$

Thus $f*e_0+e_0*g=e_0*g+f*e_0$ as desired. ∎

Ex 1 Later we will show $\pi_1(T^2, \cdot) \cong \mathbb{Z}^2$. It is generated by the loops classes $[f]$ and $[g]$ shown below



As a top. group $(0,0)$ is the identity and addition mod 1 in each coordinate is the operation. Below we illustrate a homotopy (path) between $f * g$ and $g * f$.



$f * g \simeq_p g * f$.

Ex 2 We cannot prove this here, but it is well known that

$$\pi_1\left(SL(2,\mathbb{R}), \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \cong \mathbb{Z}$$

$$\pi_1\left(SL(n,\mathbb{R}), I\right) \cong \mathbb{Z}/2\mathbb{Z} \quad \text{for } n \geq 3.$$

$SL(n,\mathbb{R})$ are $n \times n$ real matrices with $\det = 1$. The group op. is matrix multiplication, which is noncommutative in general.