§55 Retractions and Fixed Points

Def Let $A \subset X$. A retraction of X onto A is a cont. map $r: X \to A$ s.t. $r(a) = a\ \forall a \in A$.

Lemma Let $r: X \to A$ be a retraction. Let $j: A \to X$ be inclusion. Then $j_*: \pi_1(A,a) \to \pi_1(X,a)$ is one-to-one.

Pf $r \circ j = Id_A : A \to A$. By Thm 52.4 $(r \circ j)_*$ is the identity homomorphism on $\pi_1(A,a)$. Also by 52.4 $(r \circ j)_* = r_* \circ j_*$. Therefore, j_* is one-to-one. □

Thm There is no retraction from B^2 onto $S^1 = \partial B^2$.

Pf We know $\pi_1(B, (1,0))$ is trivial and that $\pi_1(S^1, (1,0)) \cong \mathbb{Z}$. Thus $j_*: \pi_1(S^1, (1,0)) \to \pi_1(B^2, (1,0))$ cannot be one-to-one. By the lemma no retraction $r: B^2 \to S^1$ can exist. □

The next result uses Thm 22.2 about q-maps, so we review it here.

Thm 22.2 Let $p: X \to Y$ be a q-map. Let $g: X \to Z$ be a map that is constant on $p^{-1}(\{y\})$, $\forall y \in Y$. Then $\exists f: Y \to Z$ s.t. $f \circ p = g$. f cont $\Leftrightarrow$ g cont. f q-map $\Leftrightarrow$ g q-map.

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "p"'] & Z \\ Y \arrow[ur, dashed, "f"'] & \end{tikzcd}$$

$f(y) = g(x)$ for any $x \in p^{-1}(y)$.

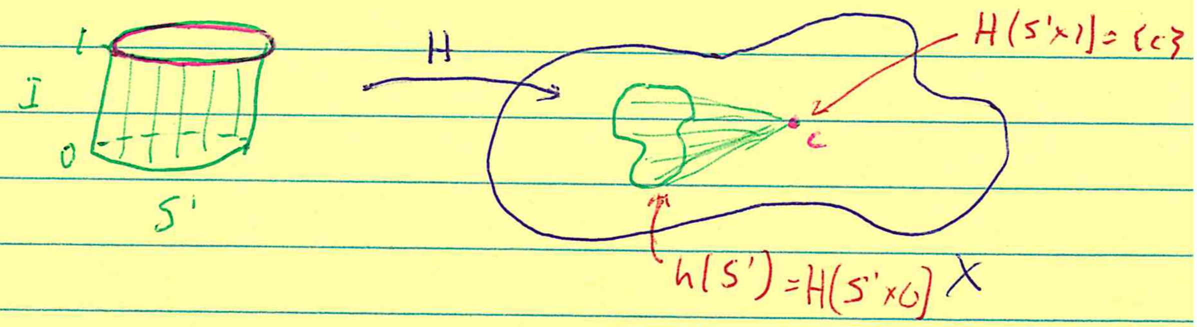
Lemma 55.3 Let $h: S^1 \to X$ be cont. The following are equivalent.

① h is nulhomotopic.

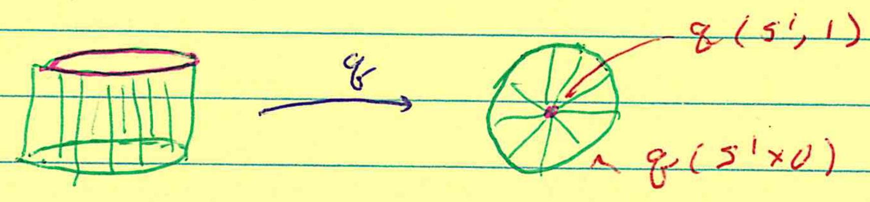
② h can be extended to a cont. map $k: B^2 \to X$.
($k(\theta, 1) = h(\theta)$.)

③ $h_*: \pi_1(S^1, b) \to \pi_1(X, x_0)$ is the trivial homomorphism
($b = (1,0)_{rect} = (0,1)_{polar}$, $x_0 = h(b)$)

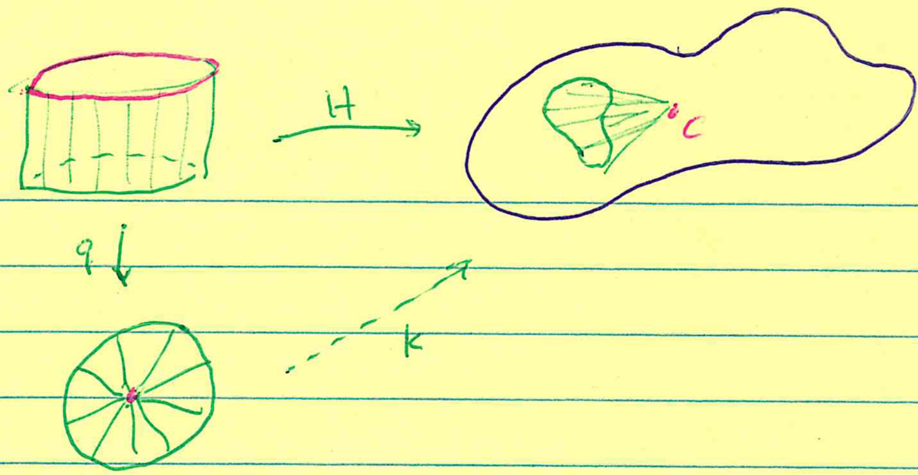
Pf ① ⇒ ② Let $H: S^1 \times I \to X$ be a homotopy from h to a constant map $C(\theta) = c \in X$.



Now let $q: S^1 \times I \to B^2$ be $q(\theta, t) = (\theta, 1-t)$ (polar)



We claim q is a q-map. Clearly q is cont and onto. If it is also a closed map then it is a q-map (see pg 137). This is easy. Since $S^1 \times I$ is compact, closed sets are compact. Hence their images in B^2 under q are compact. Since B^2 is Hausdorff compact sets are closed.



Now q is 1-to-1 everwhere except on $S^1 \times 1$ whose image is $(0,0) \in B^2$. Since $H(\theta,t)$ is always c on $S^1 \times 1$, we can apply Thm 22.2 to show $\exists$ a cont map $k: B^2 \to X$ s.t. $H = k \circ q$.

For $(\theta,1) \in \partial B^2$ we have

$$K(\theta,1) = H(\text{"}q^{-1}(\theta,1)\text{"}) = H(\theta,0) = h(\theta).$$

Thus k extends h.

(2) $\Rightarrow$ (3) Let $j: S^1 \to B^2$ be inclusion. Now $h: S^1 \to X$ extends to a cont. map $k: B^2 \to X$. Thus, $h = k \circ j$ and so $h_* = k_* \circ j_*$.

$$\begin{tikzcd} \pi_1(S^1, b) \arrow[r, "j_*"] \arrow[rr, bend right, "h_*"'] & \pi_1(B^2, b) \arrow[r, "k_*"] & \pi_1(X, x_0) \end{tikzcd}$$

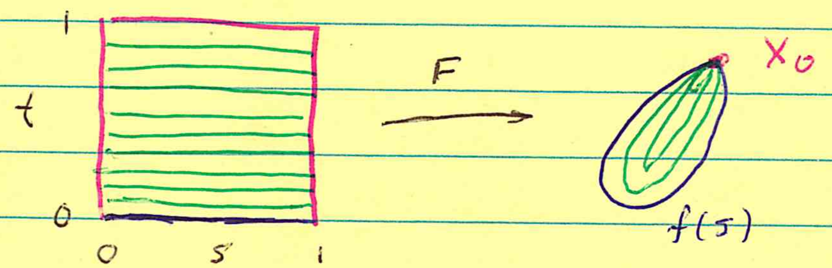
Now, since j_* is trivial, h_* is trivial

③ ⇒ ① Let $h: S^1 \to X$ be cont, $h(b) = x_0$, and suppose h_* is trivial. We will show h is homotopic to the constant map $C: S^1 \to \{x_0\}$.

Let $p: \mathbb{R} \to S^1$ be the standard covering map and let $p_0 = p|[0,1]$. Then $p_0: I \to S^1$ is a loop and $[p_0]$ generates $\pi_1(S^1, b)$.

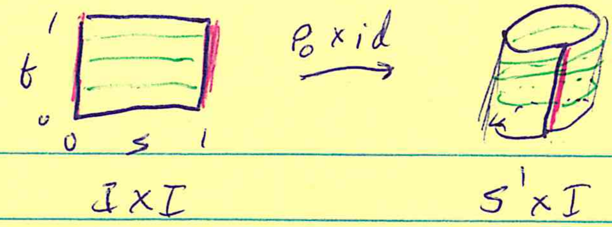
Since $h_*([p_0])$ is trivial in $\pi_1(X, x_0)$, the loop $f = h \circ p_0$ is nulhomotopic in X. Let $F: I \times I \to X$, be a path homotopy between f and e_{x_0}. We pause to study its properties.

$F(s, 0) = f(s) \quad \forall s \in I$
$F(s, 1) = e_{x_0}(s) = x_0 \quad \forall s \in I$
$F(0, t) = f(0) = e_{x_0}(0) = x_0 \quad \forall t \in I$
$F(1, t) = f(1) = e_{x_0}(1) = x_0 \quad \forall t \in I$



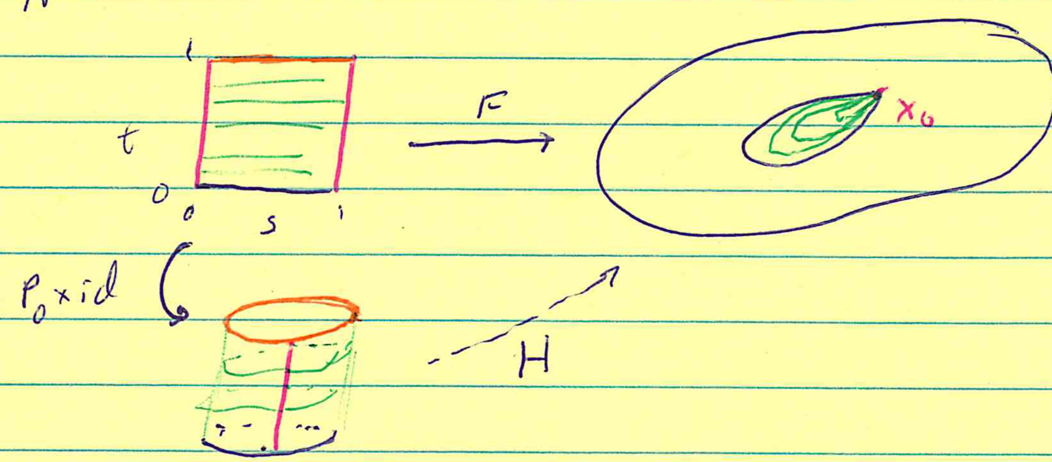
The three pink edges go to x_0.

Next, the map $p_0 \times id: I \times I \to S^1 \times I$ is a q-map that identifies $(0, t)$ with $(1, t)$, $\forall t \in I$.



The map $p_0 \times id$ is one-to-one except on $0 \times I$ and $1 \times I$, the two pink edges, where it is two-to-one.

Since F is constant along these two edges we may apply Thm 22.2



Let $H: S^1 \times I \to X$ be the map given by 22.2. We claim H is a ~~path~~ homotopy from h to $c_{x_0}(\theta) = c$.

For $\theta \in (0, 2\pi)$ we have

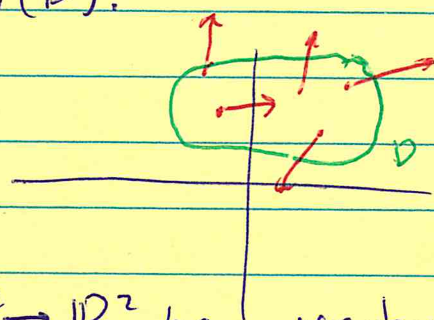
$$H(\theta, 0) = F(p_0^{-1}(\theta), 0) = f(p_0^{-1}(\theta)) = h \circ p_0 \circ p_0^{-1}(\theta) = h(\theta).$$

For $\theta = 0$, f takes either inverse image to x_0 which is $h(0)$.

$H(\theta, 1) = F(p_0^{-1}(\theta), 1) = x_0$. Thus H is a homotopy from h to a constant map, so h is nulhomotopic. $\square$

Vector Fields

Def Let $D \subset \mathbb{R}^2$. A map $v : D \to \mathbb{R}^2$ is called a *vector field*. A vector field v is *nonvanishing* if the image of v does not include the zero vector, $(0,0) \notin v(D)$.

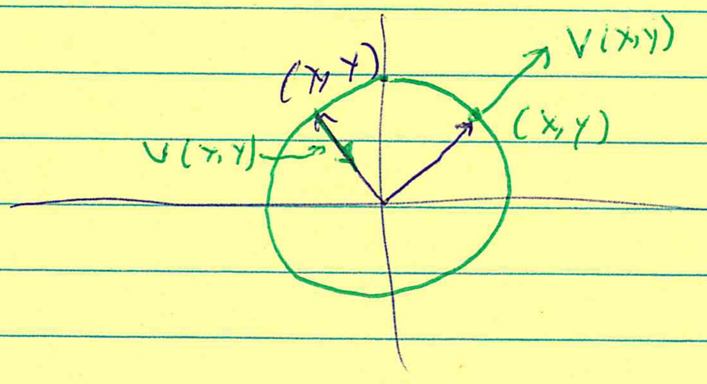


Def Let $v : B^2 \to \mathbb{R}^2$ be a vector field. For $(x,y) \in \partial B^2$, we say $v(x,y)$ *points directly outward* if

$$(x,y) + t\,v(x,y) = (0,0) \quad \text{for some } t < 0.$$

We say $v(x,y)$ *points directly inward* if

$$(x,y) + t\,v(x,y) = (0,0) \quad \text{for some } t > 0$$

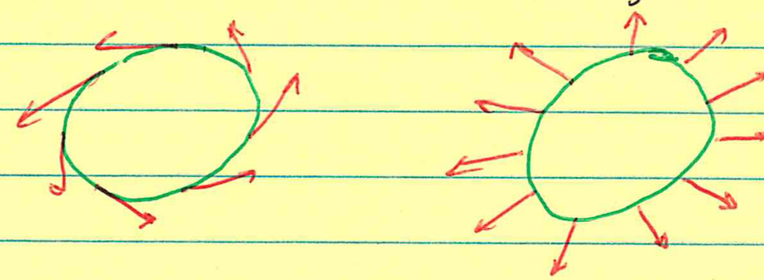


Thm 55.5 Let $v: B^2 \to \mathbb{R}^2$ be a nonvanishing continuous vector field.

(i) $\exists\, (x,y) \in \partial B^2$ where $v(x,y)$ points directly inward

(ii) $\exists\, (x,y) \in \partial B^2$ where $v(x,y)$ points directly outward.

Before studying the proof try to draw a counter-example. Consider the two figures below.



In one the vectors are tangent to ∂B^2, in the other they are $\perp$. 55.5 says if v is continuous then these two field must vanish somewhere inside B^2.

Pf Suppose there is no point on ∂B^2 where v p.d. in. Let w be the restriction of v to ∂B^2. So,

$$w : \partial B^2 \to \mathbb{R}^2 - 0$$

Since w clearly extends to $v : B^2 \to \mathbb{R}^2 - 0$, it is nulhomotopic by Lemma 55.3.

We will show however that w is homotopic to the inclusion map $j : S^1 \hookrightarrow \mathbb{R}^2 - 0$. Since j is not nulhomotopic by Cor 55.4, and $w \simeq j$, w is not nulhom. by Lemma 51.1. Hence, we have a contradiction!

Define $F : S^1 \times I \to \mathbb{R}^2 - 0$ by

$$F((x,y),t) = t\, j(x,y) + (1-t)\, w(x,y) = t(x,y) + (1-t)v(x,y).$$

We check that $F((x,y),t)$ is never $(0,0)$. This is clear for $t=0$ and $t=1$. Suppose $F((x,y),t) = 0$ for some $t \in (0,1)$. Then

$$(x,y) + \frac{1-t}{t} v(x,y) = 0$$

and $\frac{1-t}{t} \geq 0$, i.e. $v(x,y)$ is pointing directly inward!

Hence $F \neq 0$ and $w \simeq j$.

The other case is similar, just use $-v(x,y)$.

$\square$

Thm 5.5.6 (Brouwer Fixed Point Thm for the disk.)
If $f: B^2 \to B^2$ is cont. then $\exists\ (x,y) \in B^2$ s.t. $f(x,y) = (x,y)$.

Pf Suppose $f(x,y) \neq (x,y)\ \forall\ (x,y) \in B^2$. Let $h(x,y) = f(x,y) - (x,y)$. Then h is nonvanishing. Thus $\exists\ (x_0, y_0) \in \partial B^2$ where h points dir. outward. Let $w = (x_0, y_0)$. Then $\exists\ t > 0$ s.t.

$$\begin{aligned} t\, h(w) &= w \\ t\,(f(w) - w) &= w \\ t\, f(w) &= (1+t)\, w \\ f(w) &= \frac{1+t}{t}\, w \end{aligned}$$

But $\frac{1+t}{t} > 1$, so $f(w) \notin B^2$, a contradiction! $\square$

Cor Let $f: D \xrightarrow{cont} D$ where D is any top. disk. Then f has a fixed pt.

Pf Let $h: D \to B^2$ be a homeo.

$$\begin{tikzcd} D \arrow[r, "f"] \arrow[d, "h"'] & D \arrow[d, "h"] \\ B^2 \arrow[r, "g"'] & B^2 \end{tikzcd}$$

Let $g = h \circ f \circ h^{-1}$. Let $p \in B^2$ be a fixed pt of g. Let $q = h^{-1}(p) \in D$. Then

$$f(q) = h^{-1} \circ g \circ h(q) = (h^{-1} \circ g)(p) = h^{-1}(p) = q. \qquad \square$$

Cor [Perron-Frobenius Thm]. Let A be a 3×3 matrix of positive real numbers. Then A has a positive real eigenvalue, ~~with all positive e. vector.~~

Outline of proof Think of A as a linear function $\mathbb{R}^3 \to \mathbb{R}^3$.
Then the first octant is mapped into itself.
Let $B = S^2 \cap$ first octant.
It is not hard to show B is homeo to B^2.

Let $f: B \to B$ be given by $f(v) = \frac{Av}{\|Av\|}$. $\leftarrow$ never zero.

There must be a fixed pt w, $f(w) = w$.

Then $w = \frac{Aw}{\|Aw\|} \Rightarrow Aw = \|Aw\| w$.

So, w is an eigenvector with e. value $\|Aw\| > 0$.

$\square$