§56 The Fundamental Theorem of Algebra

Thm Let $p(z) = z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_1 z + a_0 \in \mathbb{C}[z]$, $n \geq 1$. Then p has a root in $\mathbb{C}$.

Pf Step 1 Regard S^1 as the unit circle in $\mathbb{C}$. Let $f: S^1 \to S^1$ be given by $f(z) = z^n$. We claim $f_*: \pi_1(S^1,1) \to \pi_1(S^1,1)$ is injective, in fact it is multiplication by n.

Let $p_0 : I \to S^1$ be $p_0(s) = e^{2\pi i s} = \cos(2\pi s) + i\sin(2\pi s)$. Then $[p_0]$ is a generator of $\pi_1(S^1,1)$.

We know $f_*([p_0]) = [f \circ p_0]$. We compute

$$f(p_0(s)) = (e^{2\pi i s})^n = e^{2\pi n i s} = \cos(2\pi n s) + i\sin(2\pi n s).$$

This loop wraps around S^1 n times. The lift to $\mathbb{R}$ is the path $s \mapsto ns$. Thus

$$[f \circ p_0] = [p_0 * p_0 * \cdots * p_0] = n[p_0].$$

Thus f_* is injective.

Step 2 Let $g: S^1 \to \mathbb{C} - 0$ be $g(z) = z^n$. We claim g is not nulhomotopic. Let $j: S^1 \to \mathbb{C} - 0$ be inclusion. Thus $g = j \circ f$, so $g_* = j_* \circ f_*$. Both j_* and f_* are one-to-one. Thus g_* is one-to-one. By Lemma 55.3 g cannot be nulhomotopic.

Step 3 Suppose $|a_{n-1}| + |a_n| + \cdots + |a_0| < 1$. Suppose $p(z)$ has no roots in B^2. Now we can regard p as the map

$$p : B^2 \to \mathbb{C} - 0.$$

Let $h = p | S^1 = \partial B^2$. By Lemma 55.3 h is nullhomotopic. But we will show $h \simeq_p g$ which we know is not nulhomotopic. This contradiction will prove that p must have a root in B^2.

Let $F : S^1 \times I \to \mathbb{C} - 0$ be given by

$$F(z,t) = z^n + t\,(a_{n-1} z^{n-1} + \cdots + a_1 z + a_0).$$

Clearly $F(z,0) = g(z)$ and $F(z,1) = h(z)$. But for F to be a valid homotopy we need to show $F(z,t)$ is never 0 (so the image is in $\mathbb{C} - 0$).

$$\begin{aligned}
|F(z,t)| &\geq |z^n| - |t\,(a_{n-1} z^{n-1} + \cdots a_1 z + a_0)| \\
&= |z^n| - t\,|a_{n-1} z^{n-1} + \cdots + a_1 z + a_0| \\
&\geq |z^n| - t\,(|a_{n-1}||z^{n-1}| + \cdots |a_1||z| + |a_0|) \\
&= 1 - t\,(|a_{n-1}| + \cdots + |a_0|) > 0.
\end{aligned}$$

Step 4 We drop the assumption on $|a_{n-1}| + \cdots + |a_0|$. But $\exists\, c > 0$ (real) s.t.

$$\left|\frac{a_{n-1}}{c}\right| + \left|\frac{a_{n-2}}{c^2}\right| + \left|\frac{a_{n-3}}{c^3}\right| + \cdots + \left|\frac{a_1}{c^{n-1}}\right| + \left|\frac{a_0}{c^n}\right| < 1.$$

Let $w = z/c$ or $z = cw$. Then

$$p(z) = p(cw) = (cw)^n + a_{n-1}\cancel{z}(cw)^{n-1} + \cdots + a_1(cw) + a_0.$$

Divide by c^n:

$$\frac{p(cw)}{c^n} = w^n + \frac{a_{n-1}}{c}w^{n-1} + \frac{a_{n-2}}{c^2}w^{n-2} + \cdots + \frac{a_1}{c^{n-1}}w + \frac{a_0}{c^n}.$$

Call this poly $f(w)$. By Step 3 $\exists\, w_0 \in B^2$ s.t. $f(w_0) = 0$. Let $z_0 = cw_0$. Then

$$p(z_0) = p(cw_0) = c^n f(w_0) = 0. \qquad \blacksquare$$