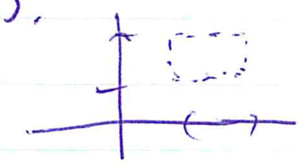
§15 Product Top.

Def Let X and Y be top spaces. Let $Z = X \times Y$.
Let $\mathcal{B} = \{U \times V \mid U \text{ is open in } X, V \text{ is open in } Y\}$.
Let $\mathcal{T}_Z$ = all unions of members of $\mathcal{B}$.
Then $\mathcal{T}_Z$ is called the <u>product top</u> for Z.



This can be generalized to finite products:
Let $Z = \prod_{i=1}^{n} X_i$, where each X_i is a top sp.
Let $\mathcal{B} = \{\prod_{i=1}^{n} U_i \mid U_i \text{ is open in } X_i\}$.

Let $\mathcal{T}_Z$ = all unions of members of $\mathcal{B}$.

Fact $\mathcal{T}_Z$ is a top.

pf Check that $\mathcal{B}$ is a basis.
(i) Let $z = (x_1, \dots, x_n) \in Z$. $\exists\, U_i \subset X_i$ open with $x_i \in U_i$, $i = 1 \dots n$.
Let $U = U_1 \times \dots \times U_n$. Then $z \in U \in \mathcal{B}$.
(ii) Let $U = \prod U_i$, $V = \prod V_i$ be in $\mathcal{B}$.
Assume $z \in U \cap V$. ($z = (z_1, \dots z_n)$)
Let $W_i = U_i \cap V_i$. No W_i is ϕ since $z_i \in U_i \cap V_i$.
Let $W = \prod W_i$. Then $z \in W \subset U \cap V$ and $W \in \mathcal{B}$.

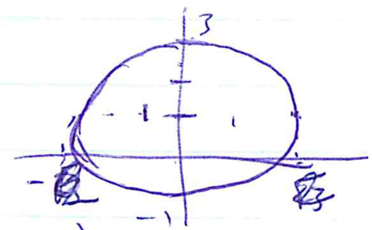
Claim For $\mathbb{R}^n$ the metric and prod. top's are the same.
You check!

Claim If $\mathcal{B}_i$ is a basis for X_i and $\mathcal{B}' = \{B_1 \times \dots \times B_n \mid B_i \in \mathcal{B}_i\}$, then $\mathcal{B}'$ generates the prod. top.

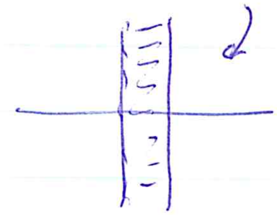
Def Let $\pi_i : \prod X_i \to X_i$ be given by $(x_1, \dots x_n) \mapsto x_i$.
These are called the <u>projection maps</u>.

Ex 1 Let $C = \{ x^2 + (y-1)^2 \le 4 \mid (x,y) \in \mathbb{R}^2 \}$.

$\pi_1(C) = [-2, 2]$
$\pi_2(C) = [-1, 3]$.



Ex 2 For $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ what is $\pi_1^{-1}((0,1])$?



$= (0,1] \times \mathbb{R}$

Ex 3 Let $\mathcal{S} = \{ \pi_i^{-1}((a,b)) \mid a < b,\ i = 1, 2 \}$.
It is a subbasis for the prod. top. on $\mathbb{R}^2$.

§16 Subspace Top.

<u>Def</u> Let $A \subset X$. Then let $\mathcal{T}_A = \{A \cap U \mid U \text{ open in } X\}$.
It is called the subsp. top. on A.

You check that it is a top.

<u>Facts</u> If $B \subset A \subset X$ then $\mathcal{T}_{B \subset X} = \mathcal{T}_{B \subset A}$.
If $B \subset A \subset X$ and A is open in X and B is open in A
then B is open in X.

<u>#3 (pg 92)</u> Let $Y = [-1, 1]$ as a subsp. of $\mathbb{R}$.
Which of the following are open in Y, in $\mathbb{R}$?

$A = (-1, -\frac{1}{2}) \cup (\frac{1}{2}, 1)$ both
$B = [-1, \frac{1}{2}) \cup (\frac{1}{2}, 1]$ Y only
$C = (-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1)$ neither
$D = [-1, -\frac{1}{2}] \cup (\frac{1}{2}, 1]$ neither
$E = \left((-1, 0) \cup (0, 1)\right) - \{\frac{1}{n} \mid n = 1, 2, 3, \ldots\}$

$= (-1, 0) \cup \left(\bigcup_{n=1}^{\infty} (\frac{1}{n+1}, \frac{1}{n}) \right)$ open in both.