# Sections 20 and 21: Metric Spaces

**Recall: Def:** $d: X \times X \longrightarrow \mathbb{R}$ is a metric if

① $d(x,y) \geq 0$, $\forall x,y \in X$ and $d(x,y)=0$ iff $x=y$.
② $d(x,y) = d(y,x)$
③ $d(x,y) + d(y,z) \geq d(x,z)$.

Open balls give a basis for a top.

**Def** Let $(X, \mathcal{T})$ be a top. sp. If $\exists$ a metric $d$ s.t. $d$ induces the topology $\mathcal{T}$, then we say $(X,\mathcal{T})$ is metrizable.

Later we will study conditions on $\mathcal{T}$ that ~~insure~~ in order for it to be metrizable.

**Def** Let $X$ be a metric sp with metric $d$. Let $A \subset X$. $A$ is bounded (wrt $d$) if $\exists M>0$ s.t.

$$d(a_1, a_2) \leq M \quad \forall a_1, a_2 \in A.$$

**Ex** Let $X = (-\frac{\pi}{2}, \frac{\pi}{2})$. Let $d(x,y) = |x-y|$ and $d'(x,y) = (\tan x, \tan y)$. You can show $d'$ is a metric. $X$ is bdd wrt $d$ but not wrt $d'$.

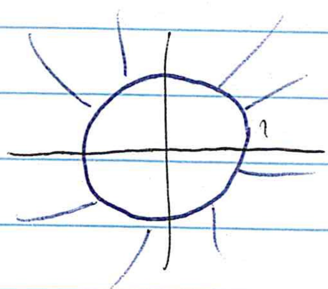
Thus boundedness is not a top. property.

**Thm 20.1** Let $(X, d)$ be a ~~m~~metric sp. Let $\bar{d} : X \times X \to \mathbb{R}$ be given by

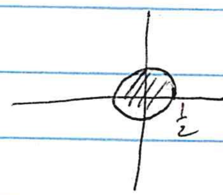
$$\bar{d}(x,y) = \min\{d(x,y), 1\}.$$

Then $\bar{d}$ is a metric on $X$ and it induces the same topology as $d$. (Yet, all sets are bdd wrt $\bar{d}$)

What do balls and circles look like if we apply this to the usual metric on $\mathbb{R}^2$?




circle of $\bar{d}$ radius 1




closed ball/disk radius $\frac{1}{2}$

circle of radius $r > 1$ is $\phi$.
ball with radius $r > 1$ is all of $\mathbb{R}^2$.

Pf Part I: $\bar{d}$ is a metric.

① If $x \neq y$ $\bar{d} = d(x,y) \neq 0$ or $1 \neq 0$.
If $x = y$ $\bar{d} = \min(0, 1) = 0$.

② $\bar{d}(y,x) = \min\{d(y,x), 1\} = \min\{d(x,y), 1\} = \bar{d}(x,y)$.

③ We need to show

$$\underbrace{\bar{d}(x,z)}_{\text{LHS}} \leq \underbrace{\bar{d}(x,y) + \bar{d}(y,z)}_{\text{RHS}}$$

Case (i) Suppose $d(x,y) \geq 1$. Then $\bar{d}(x,y) = 1$.
Since $\bar{d}(x,z) \leq 1$ we have

$$\bar{d}(x,z) \leq 1 + \bar{d}(y,z) = \bar{d}(x,y) + \bar{d}(y,z).$$

Case (ii) Suppose $d(y,z) \geq 1$. Similar to (i).

Case (iii) Both $d(x,y)$ and $d(y,z)$ are $< 1$.
Then

$$\bar{d}(x,z) \leq d(x,z) \leq d(x,y) + d(y,z) = \bar{d}(x,y) + \bar{d}(y,z).$$

Part II Show $\mathcal{T}_d = \mathcal{T}_{\bar{d}}$. We will show they have a common basis. Any metric top can be generated by using open balls of radius $\leq 1/2$. (You can prove this with $\Delta \leq$.)

Therefore $\mathcal{T}_d$ is generated by balls of $d$ radius $\leq \frac{1}{2}$ and $\mathcal{T}_{\bar{d}}$ is gen. by balls of $\bar{d}$ radius $\leq \frac{1}{2}$. But these two collections of open balls are the same.

$\boxed{/}$

**Def** Let $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. Let $\|x\| = \sqrt{\Sigma x_i^2}$.

Then $d(x,y) = \|x-y\| = \sqrt{\Sigma (x_i - y_i)^2}$ is the standard metric on $\mathbb{R}^n$. Also call the Euclidean metric.

The square metric is $\rho(x,y) = \max\{|x_i - y_i|\}_{i=1}^n$.

The proof that $\rho$ is a metric is easy. The proof that $d$ is a metric is harder. It is covered in analysis courses or see Exercise #9.

**Thm 20.3** On $\mathbb{R}^n$ the prod. top, $d$-top and $\rho$-top are the same.

We will need the following Lemma whose proof you can read.

The proof uses...

**Lemma 20.2** Let $d$ and $d'$ be two metrics on $X$. Let $\mathcal{T}$ and $\mathcal{T}'$ be the resp. top's they ~~give~~ induce. Then

$$\mathcal{T} \subset \mathcal{T}' \iff \forall x \in X, \varepsilon > 0 \ \exists \delta > 0 \text{ s.t.}$$

$$B_{d'}(x, \delta) \subset B_d(x, \varepsilon).$$

It is related to Lemma 13.3.

Pf of 20.3 Let $x, y \in \mathbb{R}^n$. Then it is easy to show

$$\rho(x,y) \leq d(x,y) \leq \sqrt{n}\, \rho(x,y).$$

Let $\varepsilon > 0$ and $x \in \mathbb{R}^n$. Then $B_d(x, \varepsilon) \subset B_\rho(x, \varepsilon)$ and so $\mathcal{T}_\rho \subset \mathcal{T}_d$. Let $\delta = \varepsilon/\sqrt{n}$. Then

$$B_\rho(x, \delta) = B_\rho\left(x, \frac{\varepsilon}{\sqrt{n}}\right) \subset B_d(x, \varepsilon).$$

Thus $\mathcal{T}_d \subset \mathcal{T}_\rho$.

We conclude $\mathcal{T}_d = \mathcal{T}_\rho$.

Now we consider the prod top. Let $B = (a_1, b_1) \times \cdots \times (a_n, b_n)$. Let $x \in B$. For each $i = 1, \ldots, n$ $\exists$ $\varepsilon_i > 0$ s.t.

$$(x_i - \varepsilon_i, x_i + \varepsilon_i) \subset (a_i, b_i).$$

Let $\varepsilon = \min\{\varepsilon_i\}$. Then $B_\rho(x, \varepsilon) \subset B$. Thus, by Lemma 13.3 $\mathcal{T}_\rho \subset \mathcal{T}_{prod}$.

Finally, let $B_\rho(x, \varepsilon)$ be a basis element of $\mathcal{T}_\rho$. Let $y \in B_\rho(x, \varepsilon)$. But $B_\rho(x, \varepsilon) = (x_1 - \varepsilon, x_1 - \varepsilon) \times \cdots \times (x_n - \varepsilon, x_n + \varepsilon)$ is itself a basis element of the prod top. Thus $\mathcal{T}_{prod} \subset \mathcal{T}_\rho$. Thus $\mathcal{T}_{prod} = \mathcal{T}_\rho$. $\blacksquare$

**Def** Consider $\mathbb{R}^J = \prod_{\alpha \in J} \mathbb{R}$. Let $x = (x_\alpha)_{\alpha \in J}$, $y = (y_\alpha)_{\alpha \in J}$, be in $\mathbb{R}^J$. Define

$$\bar{\rho}(x,y) = \sup\{\bar{d}(x_\alpha, y_\alpha) \mid \alpha \in J\}.$$

It is called the uniform metric.

**Claim** $\bar{\rho}$ is a metric. Positive definiteness and symmetry are obvious. For the triangle inequality let $x, y, z \in \mathbb{R}^J$. Then

$$\bar{\rho}(x,y) + \bar{\rho}(y,z) = \sup\{\bar{d}(x_\alpha, y_\alpha)\} + \sup\{\bar{d}(y_\alpha, z_\alpha)\}$$

$$\geq \sup\{\bar{d}(x_\alpha, y_\alpha) + \bar{d}(y_\alpha, z_\alpha)\} \quad (*)$$

$$\geq \sup\{\bar{d}(x_\alpha, z_\alpha)\} = \bar{\rho}(x,z) \quad \blacksquare$$

(*) I've put this step in an appendix.

Note: $\bar{\rho}(x,y)$ is finite since $\bar{d}$ is bounded.

**Appendix** Showing $\sup\{a_\alpha\} + \sup\{b_\alpha\} \geq \sup\{a_\alpha + b_\alpha\}$, where the sups are over $\alpha \in J$ and both $\{a_\alpha\}$ and $\{b_\alpha\}$ are bounded.

First let $A, B \subset \mathbb{R}$ be bounded and let $C = A + B = \{a+b \mid a \in A, b \in B\}$. We claim

$$\sup A + \sup B = \sup C.$$

We only need to show $\geq$ but we do $=$ for completeness. Let $x \in C$. $\exists\, a \in A, b \in B$ s.t. $x = a + b$. Thus,

$$x \leq \sup A + \sup B.$$

This gives $\sup C \leq \sup A + \sup B$.
Suppose $\sup C < \sup A + \sup B$. $\forall \varepsilon > 0$ $\exists\, a \in A, b \in B$ s.t.

$$\sup A - a < \frac{\varepsilon}{2} \text{ and } \sup B - b < \frac{\varepsilon}{2}.$$

Thus,

$$\sup A + \sup B - (a+b) < \varepsilon.$$

Choose $\varepsilon < \sup A + \sup B - \sup C$.
Then $\sup A + \sup B - (a+b) < \sup A + \sup B - \sup C$.
Thus $\sup C < a + b \in C$. This is impossible so $\sup C = \sup A + \sup B$.

Now if $D \subset C$ we know that $\sup C \geq \sup D$.

Let $A$ be the underlying set for $\{a_\alpha\}$ and
let $B$ " " " " " $\{b_\alpha\}$.

The set $\{a_\alpha + b_\alpha \mid \alpha \in J\}$ is a subset of

$\{a_\alpha + b_\beta \mid \alpha, \beta \in J\} = A + B$.

Clearly $\sup A = \sup\{a_\alpha\}$, $\sup B = \sup\{b_\alpha\}$.

Thus,

$$\sup\{a_\alpha\} + \sup\{b_\alpha\} = \sup A + B \geq \sup\{a_\alpha + b_\alpha\}.$$

$\blacksquare$

End of Appendix

**Thm 20.4** For $\mathbb{R}^J$ we have $\mathcal{T}_{prod.} \subset \mathcal{T}_{unif} \subset \mathcal{T}_{box}$.
If $J$ is infinite the inclusions are proper.

**Pf** $\mathcal{T}_{unif} \subset \mathcal{T}_{box}$ Let $x \in \mathbb{R}^J$ and $\varepsilon > 0$.
Let $B = B_{\bar\rho}(x, \varepsilon) \in \mathcal{B}_{unif}$. Let $y \in B$.
Let

$$\varepsilon_\alpha = \min\{\bar d(y_\alpha, x_\alpha - \varepsilon), \bar d(y_\alpha, x_\alpha + \varepsilon)\}.$$

Let $B' = \prod_{\alpha \in J} \left(y_\alpha - \frac{\varepsilon_\alpha}{2}, y_\alpha + \frac{\varepsilon_\alpha}{2}\right) \in \mathcal{B}_{box}$.

Then $y \in B' \subset B$. By Lemma 13.3 $\mathcal{T}_{unif} \subset \mathcal{T}_{box}$.

$\mathcal{T}_{prod} \subset \mathcal{T}_{unif}$ Let $x \in \mathbb{R}^J$. Let $B = \prod U_\alpha$
be a basic nbhd of $x$ in prod. top. Let $\alpha_1 \dots \alpha_n$
be the indices in $J$ where $U_\alpha \neq \mathbb{R}$.

For each $i = 1, .., n$ choose $\varepsilon_i \in (0, 1)$ s.t.

$$B_{\bar d}(x_{\alpha_i}, \varepsilon_{\alpha_i}) = (x_{\alpha_i} - \varepsilon_i, x_{\alpha_i} + \varepsilon_i) \subset U_{\alpha_i}.$$

Let $\varepsilon = \min\{\varepsilon_i\}$.* Then

$$x \in B_{\bar\rho}(x, \varepsilon) \subset B.$$

By Lemma 13.3 $\mathcal{T}_{prod} \subset \mathcal{T}_{box}$.

*If no such $\alpha_i$ exists, $B = \mathbb{R}^J$ and any $\varepsilon$ will work.

Now assume $J$ is infinite.

$\mathcal{T}_{prod} \neq \mathcal{T}_{unif}$. Consider $B_{\bar{\rho}}(0, \frac{1}{2}) \in \mathcal{B}_{unif}$.

For all $\alpha \in J$ the projection map gives

$$\pi_\alpha(B_{\bar{\rho}}(0, \tfrac{1}{2})) = (-\tfrac{1}{2}, \tfrac{1}{2}) \subset \mathbb{R}.$$

But for any set in $\mathcal{T}_{prod}$ all but infinitely many projections must be $\mathbb{R}$.

$\mathcal{T}_{unif} \neq \mathcal{T}_{box}$ This work is based on Exercise #6.

Suppose $J = \mathbb{N}$. Let $x = (0, 0, 0 \ldots)$, Let $U = \prod_{i=1}^{\infty} (-\frac{1}{2}, \frac{1}{2})$. It is in $\mathcal{T}_{box}$. We claim $U \notin \mathcal{T}_{unif}$.

Let $V = \mathbb{R}^\omega - U$. We will show $V$ is not closed in the unif top. Hence $U$ is not in $\mathcal{T}_{unif}$.

Consider $y = (\frac{1}{4}, \frac{3}{8}, \frac{7}{16}, \ldots, \frac{2^{n-1}-1}{2}, \ldots) \in U$.

Notice $\bar{\rho}(x, y) = \frac{1}{2}$. Thus $y \notin B_{\bar{\rho}}(x, \frac{1}{2})$.

Consider the sequence

$$\{ z_1 = (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \cdots)$$
$$z_2 = (\tfrac{1}{4}, \tfrac{1}{2}, \tfrac{1}{2}, \cdots)$$
$$z_3 = (\tfrac{1}{4}, \tfrac{3}{8}, \tfrac{1}{2}, \tfrac{1}{2} \cdots)$$
$$z_4 = (\tfrac{1}{4}, \tfrac{3}{8}, \tfrac{3}{16}, \tfrac{1}{2}, \tfrac{1}{2}, \cdots) \}$$
$$\vdots \qquad \}.$$

Each $z_i$ is in $V$ since $\frac{1}{2} \notin (\frac{-1}{2}, \frac{1}{2})$.
Now $\bar{\rho}(z_i, y) = \frac{1}{2^{i+1}} \to 0$. Thus $z_i \to y$ in

the unif. top. Hence $V$ does not contain all its limits points and so is not closed.

If $J$ is uncountable the same construction can be done using any countable subset of $J$ with an arbitrary ordering. $\boxed{!}$

We will show shorty that $\mathbb{R}^\omega$ is not metrizable in the box top. Thm 20.5 shows that the product top. on $\mathbb{R}^\omega$ is metrizable. The metric, or a metric rather, is

$$D(x,y) = \sup_{i=1,2,\dots} \left\{ \frac{\bar{d}(x_i, y_i)}{i} \right\}.$$

Study the Proof given in the text book.

§21 **More on metric spaces.**

Some quick facts you can check.

(a) Metric spaces are Hausdorff.

(b) Subspaces of metric spaces are metric spaces.

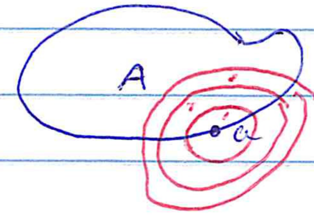
(c) Countable products of metrizable spaces are metrizable. [See Exercise #3 in §21. It generalizes Thm 20.5.]

(d) The $\varepsilon$-$\delta$ def. of continuity in a metric spaces is equivalent to the topological def. We did this.

**Lemma 21.2** (The Sequence Lemma)
Let $X$ be a top. sp. with $A \subset X$. Suppose $\{a_n\}_{n=1}^{\infty} \subset A$ and $a_n \to a$. Then $a \in \bar{A}$.
If $X$ is metrizable, the converse holds:
$a \in \bar{A} \Rightarrow \exists \{a_n\} \subset A$ with $a_n \to a$.

**Pf** By def. of a convergent seq. every nbhd $U$ of $a$ contains $a_n$ for $n$ big enough. Thus, every nbhd of $a$ meets $A$. By Thm 17.5 $a \in \bar{A}$.

Assume $X$ is metrizable. Let $a \in \bar{A}$. We will find a seq $\{a_n\} \subset A$ s.t. $a_n \to a$. For each $n \in \mathbb{N}$, consider the open ball $B(a, \frac{1}{n})$. Pick $a_n \in B(a, \frac{1}{n}) \cap A \neq \emptyset$. You can check that $a_n \to a$. $\square$

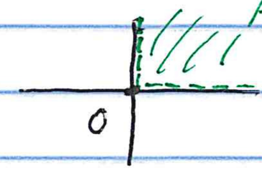



**Thm 21.3** Let $f: X \to Y$. If $f$ is cont. and $x_n \to x$ in $X$, then $f(x_n) \to f(x)$ in $Y$. The converse holds when $X$ is metrizable.

**Pf** See textbook.

**Ex 1** $\mathbb{R}^\omega$ in the box top is not metrizable.

**Pf** We will find a contradiction will the Seq. Lemma. Let $\mathbf{0} = \{0, 0, 0, \dots\}$ and $A = \{(x_1, x_2, x_3 \dots) \mid x_i > 0 \ \forall i \in \mathbb{N}\}$. We will show that $\mathbf{0} \in \bar{A}$, but no seq. in $A$ converges to $\mathbf{0}$.




Let $U$ be a nbhd of $\mathbf{0}$. Then $\exists$ a basis element $B$ s.t. $\mathbf{0} \in B \subset U$. It must be of the form

$$B = (a_1, b_1) \times (a_2, b_2) \times (a_3, b_3) \times \cdots$$

where $a_i < 0 < b_i$, for each $i$. Then $\left(\frac{b_i}{2}\right)_{i=1}^{\infty}$ is in $B$ and $A$. Thus $U \cap A \neq \phi$. Hence $\mathbf{0} \in \bar{A}$.

Now, let $(a_n)_{n=1}^{\infty} \subset A$ where $a_n = (a_{n1}, a_{n2}, a_{n3}, \dots)$ with all $a_{ij} > 0$. Consider the basis member

$$B' = \left(-\frac{a_{11}}{2}, \frac{a_{11}}{2}\right) \times \left(-\frac{a_{22}}{2}, \frac{a_{22}}{2}\right) \times \left(-\frac{a_{33}}{2}, \frac{a_{33}}{2}\right) \times \cdots$$

Clearly $\mathbf{0} \in B'$. But $a_1 \notin B'$ since $a_{11} \notin \left(-\frac{a_{11}}{2}, \frac{a_{11}}{2}\right)$, $a_2 \notin B'$ since $a_{22} \notin \left(-\frac{a_{22}}{2}, \frac{a_{22}}{2}\right)$, and so on.

In general $a_n \in B'$ since $a_{nn} \notin \left(-\frac{a_{nn}}{2}, \frac{a_{nn}}{2}\right)$, $\forall n$. There for it is impossible for $a_n \to 0$ for any seq. in $A$. By the Seq. Lemma $A$ is not metrizable.

**Ex 2** If $J$ is uncountable, then $\mathbb{R}^J$ is not metrizable even in the prod. top. See textbook for proof.

# Summary

| $J$ | $\mathbb{R}^J$ |
|---|---|
| $\{1, 2, 3, \ldots, n\}$ | product, euclidean, box topologies are the same |
| Infinite | prod. top. $\subsetneq$ unif. top. $\subsetneq$ box top. |
| Infinite, countable | prod. top. metrizable<br>box top. not metrizable |
| Uncountable | prod. top not metrizable |

**Note** See the Supplementary Exercises: Nets, pages 187-188, especially #6. Nets are a generalization of sequences. They behave "better" than sequences: If a net in a subset $A \subset X$ converges to $a$, then $a \in \bar{A}$, and the converse holds even when $X$ is not metrizable.

# Uniform Convergence

**Def** Let $X, Y$ be top. sp's with $Y$ a metric sp. with metric $d$. Let $f_n : X \to Y$, $n = 1, 2, \ldots$, be a seq. of functions. We say $f_n \to f$ uniformly if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } n \geq N \Rightarrow d(f_n(x), f(x)) < \varepsilon \quad \forall x \in X.$$

The point is $N$ does not depend on $x$; one $N$ works for all $x \in X$. Some texts use the notation $f_n \rightrightarrows f$.

Point-wise convergence, $f_n(x) \to f(x)$ $\forall x \in X$, is weaker than uniform convergence.

**Ex** Let $f_n : [0,1] \to \mathbb{R}$ be given by $f_n(x) = x^n$. Then

$$f_n(x) \to f(x) = \begin{cases} 0 & x < 1, \\ 1 & x = 1. \end{cases}$$

converges pointwise but not uniformly.

**Thm** 21.6 (Unif Limit Thm) Let $X$ be a top sp and $Y$ be a metric sp. Let $f_n : X \to Y$, $n \in \mathbb{N}$, be a seq. of continuous functions. If $f_n \to f$ unif., then $f$ is cont.

**Pf** See textbook.