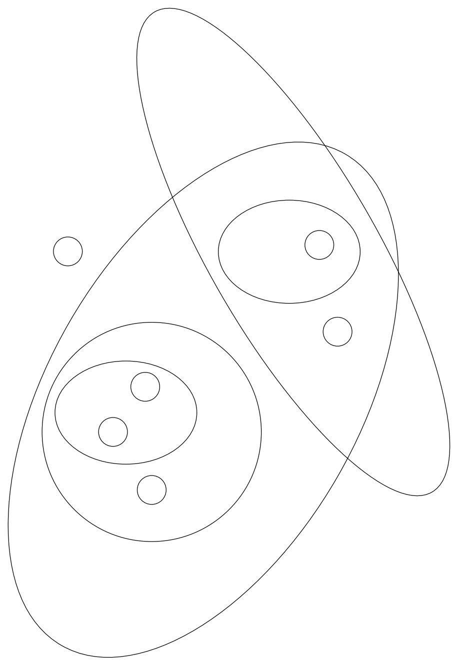
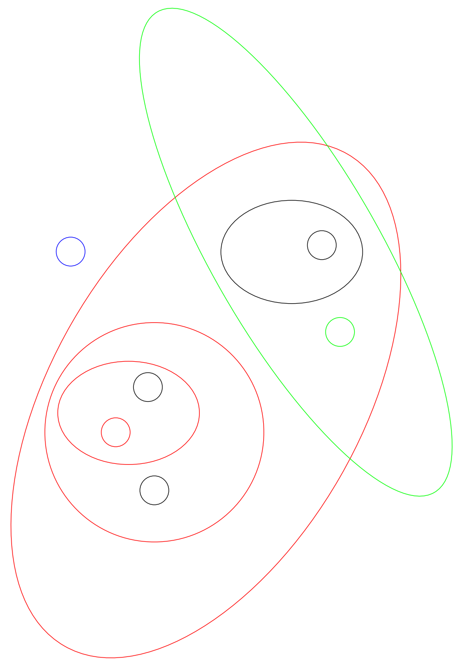
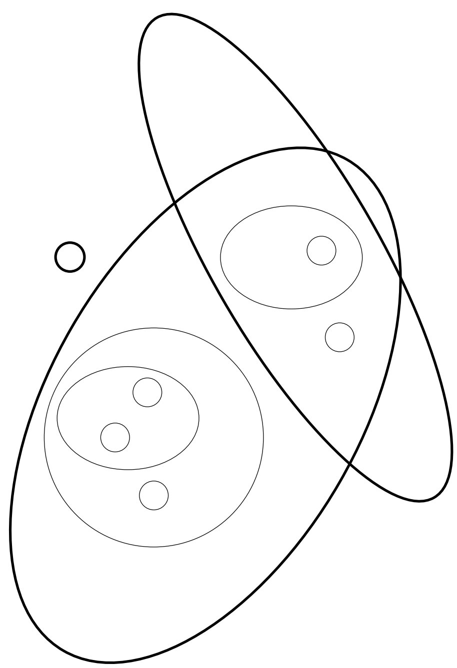
Illustrating The Maximal Principle and Zorn's Lemma

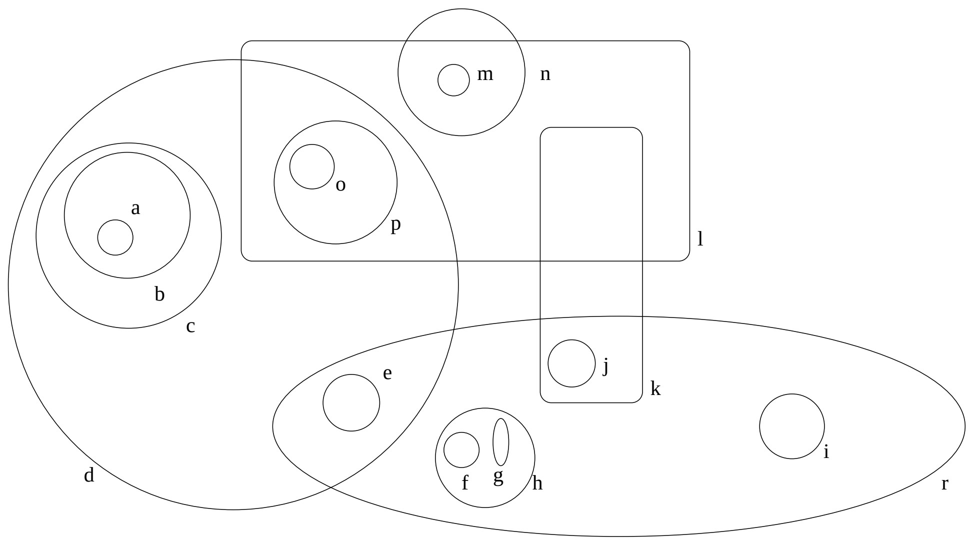
Example 1. Below in the upper left is a collection of subsets of the plane. They are strictly partially ordered by proper inclusion. In the upper right the colors indicate three maximal simply ordered subcollections; the one in red has four members, the one in green has two members and the blue one has only one member. There are three additional maximal simply ordered collections; find them. In the figure below the three the darkened sets are the three maximal members.





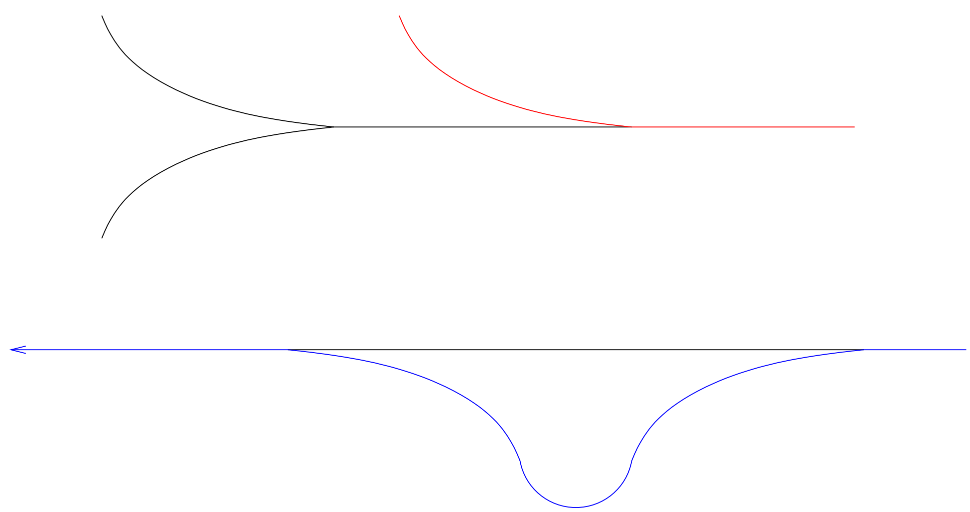


Exercise. Using proper set inclusion for a strict partial order on the subsets of the plane shown below list all the maximal simply ordered subcollections. List the maximal sets.

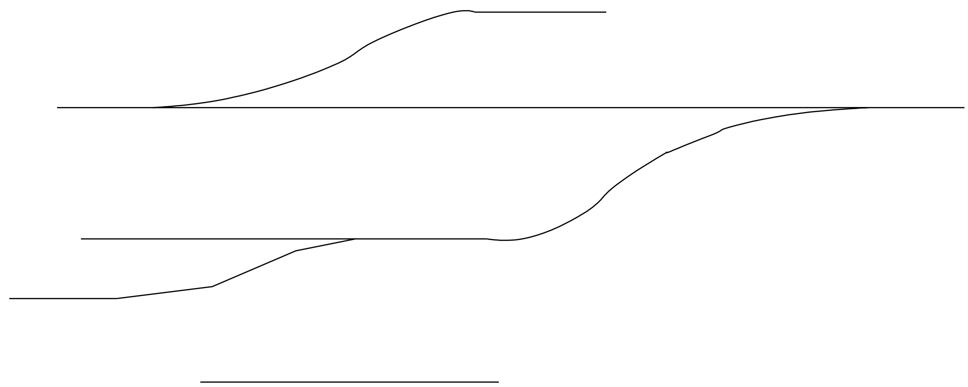


Example. In the graph G shown below, for any two points a and b in G, we define $a < b$ if and only if there is a path in G from a to b such that in traveling from a to b one always moves rightward, never backing up. If there is no such path from a to b or from b to a, then a and b are not comparable. This gives a strict partial order on G. It is given that G is a closed set in the plane. Note that G is not connected.

We have marked two maximal simply ordered subsets of G, one is red, the other is blue. There are three others; find them. Since each simply ordered subset of G has a maximal member, Zorn's Lemma tells us that there exists a maximal member of G. In this example there are two maximal members.



Exercise. In the graph use the same strict partial order as in the example above. Use color pens to indicate all the the maximal simply ordered subsets. Mark the maximal points. Assume the graph is closed.



Exercise. Use the usual less than relation $<$ on $[0, \infty)$ as a strict partial order. Explain why Zorn's does not apply. Consider the collection of all disks in the plane with proper set inclusion as a strict partial order. Explain why Zorn's does not apply.

Note. In the examples and exercises here we could show when maximal members exist without using Zorn's lemma. Zorn's lemma is useful in settings that are so complex that maximal members cannot be found by simple inspection.