Classification of homogeneous continua in the plane

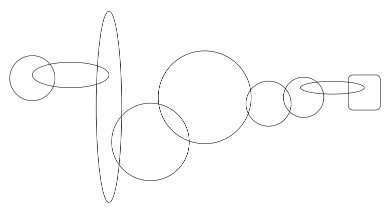
Notes for Math 530

A **continuum** is a connected, compact, metric space. A continuum is **non-degenerate** if it has more than one point. In the plane a closed arc, a circle, and the topologist's sine curve are examples. A topological space X is **homogeneous** if for every $a, b \in X$, there is a homeomorphism, $h : X \to X$ with $h(a) = b$. From the above examples only the circle is homogeneous.

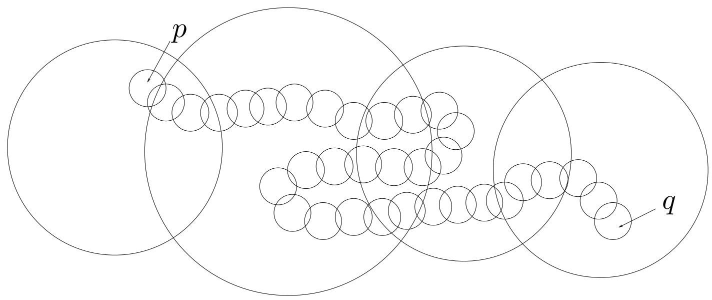
In 1920 Knaster and Kuratowski [4] asked whether the circle was the only non-degenerate homogeneous continuum in the plane. In 1948 Bing [1] showed that the **pseudo-arc** was another such space and in 1959 Bing and Jones [2] showed that the **circle of pseudo-arcs** was a third such space. Any two pseudo-arcs or circles of pseudo-arcs are respectively homeomorphic, just as any two simple closed curves are. The circle, the pseudo-arc and the circle of pseudo-arcs are topologically distinct spaces.

In 2014 Hoehn and Oversteegen [3] showed that up to homeomorphism these are the only three examples, thus giving a complete topological classification of non-degenerate, homogeneous, planar continua.

Here is a construction of the pseudo-arc. All sets are in the plane. A **chain** is an ordered finite collection of open sets, $C = (L_1, L_2, \ldots, L_n)$, such that $L_i \cap L_j \neq \emptyset$ iff $|i - j| \leq 1$. The members of a chain are called its **links** and if each link has diameter less than some $\epsilon > 0$ it is called an ϵ-chain.



Let $C = (L_1, \ldots, L_n)$ and $C' = (L'_1, \ldots, L'_{n'})$ be chains such that each link of C is a subset of some link of C'. We say C is **crooked** in C' if for all indices i, j, m and n with $L_i \cap L'_m \neq \emptyset$, $L_j \cap L'_n \neq \emptyset$ and $m < n - 2$, there exists indices k and l such that $i < k < l < j$ or $i > k > l > j$, $L_k \subset L'_{n-1}$, and $L_l \subset L'_{m+1}$.



A pseudo-arc is then constructed as follows. Let $p, q \in \mathbb{R}^2$, $p \neq q$. For $i = 1, 2, 3\ldots$ let $C^i = (L^i_1, \ldots, L^i_{n_i})$ be chains such that

- $p \in L^i_1$ and $q \in L^i_{n_i}$,
- C^i is an ϵ-chain with $\epsilon = 1/2^i$,
- the closure of each link of C^{i+1} is a subset of some link of C^i, and
- C^{i+1} is crooked in C^i.

Let $P = \bigcap_{i=1}^{\infty} \left(\bigcup_{j=1}^{n_i} L_j^i \right)$. Then P is a pseudo-arc.

Remark. A similar definition can be given for the **pseudo-circle** (each last link meets the first and $p = q$). However, is has been shown that the pseudo-circle is not homogeneous. [5]

Finally we discuss the circle of pseudo-arcs. A space X is a circle of pseudo-arcs if there exists an open continuous function $f : X \to S^1$ such that $f^{-1}(\theta) \approx P$ for every $\theta \in S^1$. Bing and Jones [2] showed these have the required properties and constructed a circle of pseudo-arcs in the plane.

References

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