Urysohn's Lemma

Urysohn's Lemma. Let X be a normal space. Let A and B be disjoint closed subsets of X. Then,

$$\exists f : X \xrightarrow{\text{cont}} [0,1] \text{ such that } f|A = 0 \text{ and } f|B = 1.$$

(We sometimes say A and B can be separated by a continuous function.)

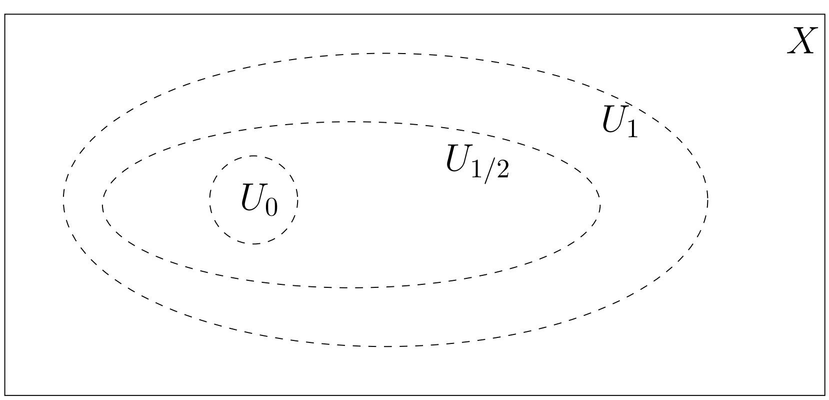
Definition. Let $\mathbb{D} = \left\{ r = \dfrac{m}{2^n} \,\middle|\, 0 \leq m \leq 2^n \,,\, n \geq 0 \right\}$. This set is called the *dyadic rationals* in [0,1]. It is a countable dense subset of [0,1].

Definition. Let X be a topological space. By a *Urysohn family* we mean a collection of open sets indexed by $\mathbb{D}$,

$$\mathcal{U} = \{U_r \,|\, r \in \mathbb{D}\}, \text{ such that } s < t \Rightarrow \overline{U}_s \subset U_t.$$

We define the *associated Urysohn function* $f = f_{\mathcal{U}} : X \to [0,1]$ by

$$f(x) = \begin{cases} \inf\{r \in \mathbb{D} \,|\, x \in U_r\} & \text{if } x \in U_1, \\ 1 & \text{otherwise.} \end{cases}$$



Lemma. Associated Urysohn functions are continuous.

Proof. Let f be the Urysohn function associated to a Urysohn family $\mathcal{U} = \{U_r\}$ for a space X. We will show that for any $s \in (0,1)$ that $f^{-1}[0,s)$ and $f^{-1}(s,1]$ are open. It then follows that f is continuous since these intervals form a subbasis of $[0,1]$ in the usual subspace topology (see page 103 of Munkres). To this end we observe that

$$f(x) < r \;\Rightarrow\; x \in U_r, \text{ thus} \tag{1}$$
$$x \notin U_r \;\Rightarrow\; f(x) \geq r; \text{ and} \tag{2}$$
$$x \in U_r \;\Rightarrow\; f(x) \leq r, \text{ thus} \tag{3}$$
$$f(x) > r \;\Rightarrow\; x \in X - U_r. \tag{4}$$

We will show that $f^{-1}[0,s) = \bigcup\{U_r \,|\, r < s,\ r \in \mathbb{D}\}$, which is open. Let $x \in U_r$ for some $r \in \mathbb{D}$ with $r < s$. By (3), $f(x) \leq r < s$. Thus, $f(x) \in [0,s)$ which is to say $x \in f^{-1}[0,s)$. Hence each $U_r \subset f^{-1}[0,s)$ for $r < s$. This gives one inclusion.

For the other inclusion, suppose $x \in f^{-1}[0,s)$. Then $f(x) < s$ and there exists a $t \in \mathbb{D}$ such that $f(x) < t < s$, since $\mathbb{D}$ is dense. Then by (1), $x \in U_t \subset \bigcup\{U_r \,|\, r < s\}$.

Similarly, we can show $f^{-1}(s,1] = \bigcup\{X - U_r \,|\, r > s\}$, but it is not obvious this is open. Instead we will show that $f^{-1}(s,1] = \bigcup\{X - \overline{U}_r \,|\, r > s\}$, which is open.

Suppose, $x \in \bigcup\{X - \overline{U}_r \,|\, r > s\}$. Then for some $r > s$,

$$x \in X - \overline{U}_r \;\Rightarrow$$
$$x \notin U_r \;\Rightarrow$$
$$f(x) \geq r > s \;\Rightarrow$$
$$x \in f^{-1}(s, 1].$$

Now suppose $x \in f^{-1}(s, 1]$. As noted $f^{-1}(s, 1] = \bigcup\{X - U_r \,|\, r > s\}$. Hence $x \in \bigcup\{X - U_r \,|\, r > s\}$. Then for some $t \in \mathbb{D}$ with $t > s$ we have $x \in X - U_t$. Let $q \in \mathbb{D}$ be such that $s < q < t$. Then

$$\overline{U}_q \subset U_t \;\Rightarrow$$
$$X - U_t \subset X - \overline{U}_q \;\Rightarrow$$
$$x \in \bigcup\{X - \overline{U}_r \,|\, r > s\}.$$

□

Proof of Urysohn's Lemma. We shall construct a Urysohn family of open sets $\mathcal{U}$ such that the associated function f satisfies the desired conclusion.

Let $U_1 = X - B$. Notice $A \subset U_1$, with A closed and U_1 open.

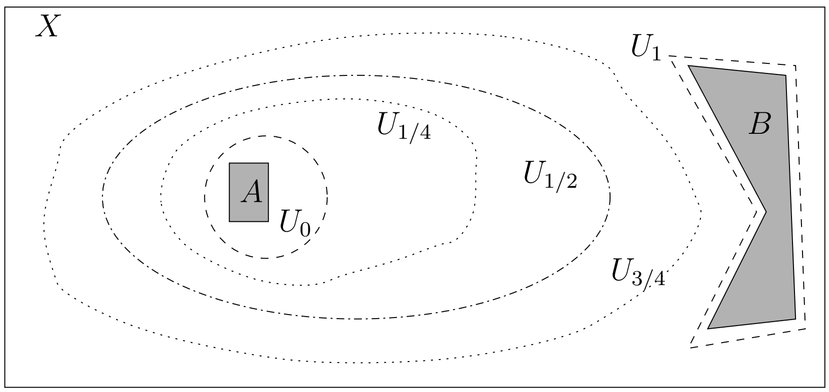
By normality and Lemma 31.1(b) there exists an open set V such that $A \subset V$ and $\bar{V} \subset U_1$. Let $U_0 = V$.

Notice $\bar{U}_0 \subset U_1$. Thus there exists an open set $U_{1/2}$ such that $\bar{U}_0 \subset U_{1/2}$ and $\bar{U}_{1/2} \subset U_1$.

There exists an open set $U_{1/4}$ such that $\bar{U}_0 \subset U_{1/4}$ and $\bar{U}_{1/4} \subset U_{1/2}$; and there exists an open set $U_{3/4}$ such that $\bar{U}_{1/2} \subset U_{3/4}$ and $\bar{U}_{3/4} \subset U_1$.

We thus define a U_r for each $r \in \mathbb{D}$ recursively (see Section 8 of Munkres).

The function f associated to $\mathcal{U} = \{U_r\}$ is zero on A since $A \subset U_0$, and one on B since f is one off of U_1 and $B = X - U_1$. □



Acknowledgment. These notes follow the proof strategy used in Bredon's *Topology and Geometry*, Spring-Verlag, New York, 1993; see pages 29–30.