π_n

These are just some rough remarks on the higher homotopy groups. This won't be on the final.

$\boxed{\pi_0}$

$\pi_0(X, x)$ exists but is not a group. It is defined as follows. Notice that $S^1 = \partial D^2$, $S^2 = \partial B^3$ and in general
$\underset{\text{disk}}{\uparrow}$ $\underset{\text{ball}}{\uparrow}$

$$S^n = \{(x_1, \dots x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots x_{n+1}^2 = 1\}$$
$$= \partial B^{n+1}$$

We define $S^0 = \partial I = \{0, 1\}$. Then

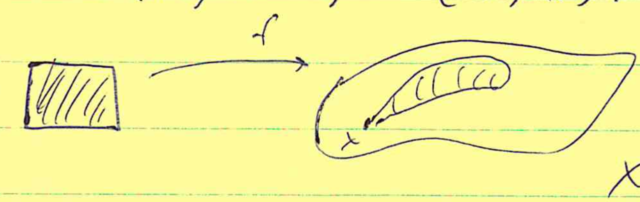
$$\pi_0(X, x) = \{f : \{0,1\} \to X \mid f(0) = x\} / \text{homotopy}.$$

If X is path conn'd π_0 has one element.
In general $|\pi_0| = \#$ path components of X.

$\pi_0(X, x)$ is not given a group structure. There is not a good way to define path addition. So $\pi_0(X, x)$ is just a set.

$\boxed{\pi_2}$ Instead of loops we use "baloons."

Let $f : (I^2, \partial I^2) \to (X, x)$ be cont.



The entire boundary of I^2 gets mapped to the base point x.

Let f and g be two such "baloons" based at the same pt $x \in X$. Then $f \simeq g$ rel ∂I^2 is our eq. rel.

$$\pi_2(X, x) = \{ [f] \mid f : (I^2, \partial I^2) \xrightarrow{\text{cont}} (X, x) \}.$$

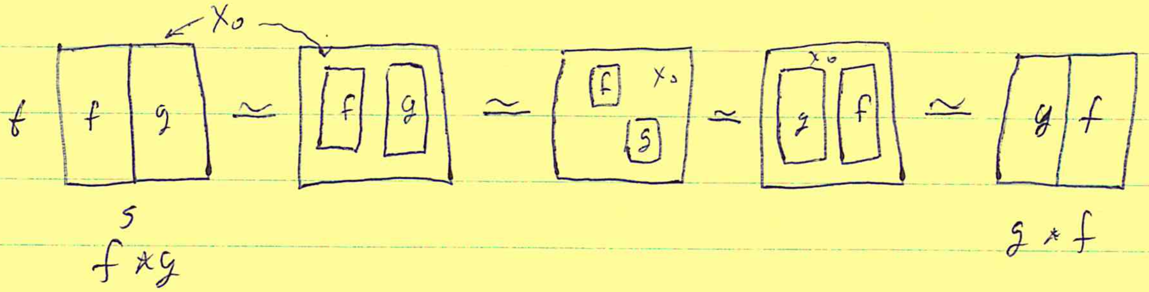
The group operation is $[f] * [g] = [f * g]$ where

$$(f * g)(s, t) = \begin{cases} f(2s, t) & s \in [0, \tfrac{1}{2}],\ t \in [0,1] \\ g(2s, t) & s \in [\tfrac{1}{2}, 1],\ t \in [0,1] \end{cases}$$

One checks this is well defined.

Thm $\pi_2(X, x)$ is abelian.

Idea of Pf



Examples $\pi_2(\mathbb{R}^3) = 0$, $\pi_2(B^3) = 0$, $\pi_2(S^3) = 0$

$\pi_2(S^3 - \text{one pt}) = 0$, $\pi_2(S^2 - \text{two pts}) \cong \mathbb{Z}$.

$\pi_2(S^2 - \text{knot}) = 0$ $\quad \pi_2(S^2 - \text{two knots}) = \begin{cases} 0 & \text{if they are linked} \\ \mathbb{Z} & \text{if not!} \end{cases}$

$\boxed{\pi_n}$ $\pi_n(X, *)$ is defined similarly.

$$\pi_n(X, x) = \{ [f] \mid f : (I^n, \partial I^n) \xrightarrow{\text{cont}} (X, x) \}$$

Facts If X is contractable $\pi_n(X,x)=0$, $\forall n\geq 1$.

If there is a path from a to b in X then

$$\pi_n(X,a)\cong\pi_n(X,b)\quad\forall n$$

If X and Y are path connected

$$\pi_n(X\times Y, (x,y))\cong\pi_n(X,x)\oplus\pi_n(Y,y)$$

Covering Spaces These behave differently for $n\geq 2$. Recall in Munkres we had

Thm 54.6 Let $p:E\to B$ be a covering map with $p(e_0)=b_0$. Then $p_*:\pi_1(E,e_0)\to\pi_1(B,b_0)$ is a monomorphism.

New Thm (Prop 4.1 in Hatcher) For $n\geq 2$, p_* is an isomorphism.

This is basically b/c S^n is simply connected for $n\geq 2$.

Fact $\pi_n(S^1)=0$ for $n\geq 2$.

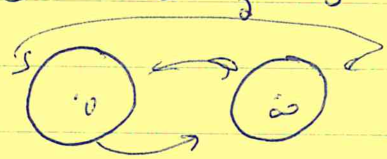
pf Let $p:\mathbb{R}\to S^1$ be the standard covering map.
Since $\mathbb{R}$ is contractable $\pi_n(\mathbb{R})=0$ $\forall n\geq 1$.
Thus for $n\geq 2$ Prop 4.1 shows $\pi_n(S^1)\cong\pi_n(\mathbb{R})=0$.

Fact $\pi_n(T^k)=0$, $n\geq 2$.

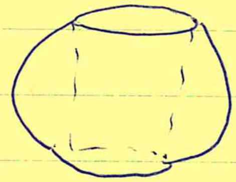
Ex $\pi_3(S^2) \cong \mathbb{Z}$.

I'll draw a map from S^3 to S^2 that can be used as a generator. First, S^3 can be modeled as the union of two solid tori.

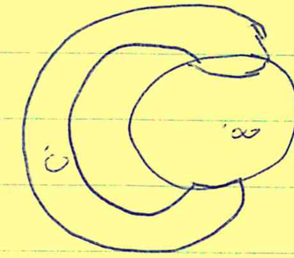
Start with $S^3 = B^3 \cup_{\partial} B^3$



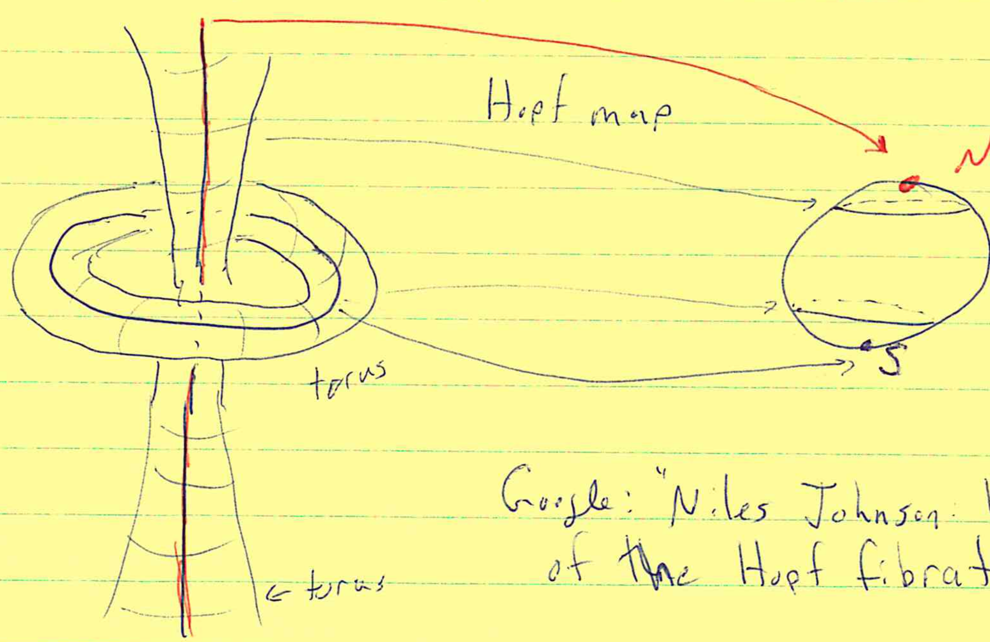
Drill out a core of one of these balls



and glue it to the other



Now we have two solid tori.



Google: "Niles Johnson: Visualizations of the Hopf fibration."

It gets worse. The table below is from Hatcher's book Algebraic Topology (Google it).

$\pi_i(S^n)$

$n \downarrow$ / $i \rightarrow$	1	2	3	4	5	6	7	8	9	10	11	12
1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0
2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_{12}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_{24} \times \mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$
5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$
6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$
7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0
8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0

There are some obvious paterns. $\pi_n(S^n) \cong \mathbb{Z}$. $\pi_i(S^n) = 0$ for $i < n$. But, while much else is known, the general question of how to compute $\pi_i(S^n)$ for all cases is unsolved.