§ 38 Stone-Čech Compactification.

Def (From §33) A space X is <u>completely regular</u> if
- one pt sets are closed and
- $\forall x \in X$, $C \subset X$ closed, $x \notin C$
  $\exists\, f: X \xrightarrow{\text{cont}} [0,1]$ s.t.
  $f(x) = 1$ and
  $f(C) = \{0\}$.

Fact Normal $\underset{\text{(U. Lem)}}{\Rightarrow}$ comp. reg. $\underset{\text{(pg 211)}}{\Rightarrow}$ reg

Recall: Def Let X be a top. sp. A <u>compactification</u> of X is a compact Hausdorff sp. Y that contains X as a subsp. s.t. $\bar{X} = Y$.

Two comp...tions Y_1, Y_2 are <u>equivalent</u> if $\exists$ homeo $h: Y_1 \to Y_2$ s.t. $h|X = \text{Id}$.

Fact X must be comp. reg. to have a comp...tion. Pf:
Y comp. H. $\underset{\text{Thm 32.3}}{\Rightarrow}$ Y normal $\Rightarrow$ Y comp. reg. $\underset{\text{Thm 33.2}}{\Rightarrow}$ X is comp. reg.

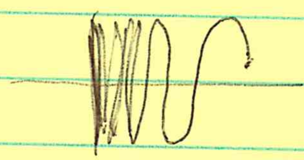
Basic Problem: When can a real valued cont. functions on X be extended continuously (uniquely?) to a comp...tion of X?

$f: X \to \mathbb{R}$

Ex $X = (0,1) \longrightarrow \overset{\infty}{O}$ Need $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 1} f(x)$

$X = (0,1) \longrightarrow [0,1]$. Need limit to exist.

What if we wanted to extend a function like
$f(x) = \sin(\frac{1}{x})$?



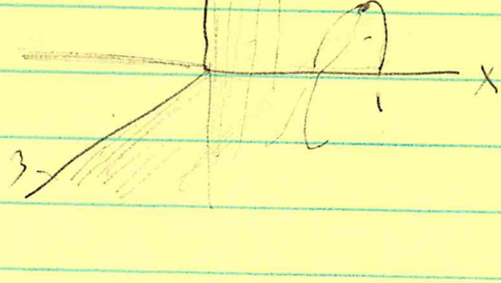
Show $\overline{S}$ is a cptification of $(0,1)$.

$$\text{Let } Y = \overline{\{(x, \sin\tfrac{1}{x}) \mid x \in (0,1)\}} \leftarrow h(X)$$

Define $F: Y \to \mathbb{R}$ as follows. Let $p \in Y$.
Let $i(p) = (x,y) \in \mathbb{R}^2$ be inclusion
$F(p) = \pi_2(i(p)) = \pi_2(x,y) = y$.
Then if $p \in h(X)$, $F(p) = \sin\frac{1}{x} = f(x)$.

Can we find a compactification that works for all bounded cont. fncs?

Suppose we want to extend $\sin(\frac{1}{x})$ and $3\sin(\frac{1}{x^2})$?
Consider $I \times I \times [-3,3]$.



$\mathbb{R}^2 \times X$

<u>Lemma 38.1</u> Let $h: X \to Z$ be an <u>embedding</u>, where Z is a compact Hausdorff space. Then $\overline{h(X)}$ is a compactification of $h(X)$ which we can also think of as a compactification of X.

<u>Lemma 38.3</u> Let $A \subset X$. Let $f: A \to Z$ be cont. with Z Haus. There is at most one extention of f to a cont func. $g: \bar{A} \to Z$.

<u>Pf</u> Suppose g, g' are two such extentions. Suppose $\exists x \in \bar{A}$ with $g(x) \neq g'(x)$. Clearly $x \in \bar{A} - A$. Let U, U' be resp. disjoint nbhds of $g(x)$ and $g'(x)$ in Z. By cont $\exists W, W'$ nbhds of x in $\bar{A}$ s.t. $g(W) \subset U$, $g(W') \subset U'$. Let $V = W \cap W'$. Then $x \in V$ and $g(V) \subset U$, $g(V') \subset U'$. We know $A \cap V \neq \phi$ so let $q \in A \cap V$. Then $g(q) = f(q) \in U$, and $g'(q) = f(q) \in U'$ so $f(q) \in U \cap U' = \phi$. Contradiction. Thus $g = g'$.

<u>Thm 34.2</u> Let X be a space in which one point sets are closed. Suppose that $\{f_\alpha\}_{\alpha \in J}$ is an indexed family of cont. funcs, $f_\alpha : X \to [0,1]$ s.t. $\forall x_0 \in X$ and $\forall$ nbhd U_0 of x $\exists \alpha$ s.t.

$$f_\alpha(x_0) > 0 \quad \text{and} \quad f_\alpha(X - U) = \{0\},$$

Then the function $F: X \to \mathbb{R}^J$ defined by

$$F(x) = (f_\alpha(x))_{\alpha \in J}$$

is an embedding of X into $\mathbb{R}^J$.

It follows that a space is completely reg. iff it it homeo to a subspace of $[0,1]^J$, for some J.

Thm 38.2 Let X be a completely regular space. $\exists$ a compactification of Y of X s.t. every bounded cont. $f: X \to \mathbb{R}$ extends (uniquely) to a cont. map $g: Y \to \mathbb{R}$.

Pf Uniqueness comes from 2nd Lemma.

Let $\{f_\alpha\}_{\alpha \in J}$ be the family of all bdd cont func's $X \to \mathbb{R}$.

$\forall \alpha \in J$ let $I_\alpha = [\inf f_\alpha(X), \sup f_\alpha(X)]$. Let $Z = \prod_{\alpha \in J} I_\alpha$.

Define $h: X \to Z$ by $h(x) = (f_\alpha(x))_{\alpha \in J}$.

By The Tych. thm Z is compact.

By Thm 34.2 h is an embedding.

By Lemma 38.1 $Y = \overline{h(X)} = X$ can be taken as a comp$\cdots$tion of X. Let $i: Y \to Z$ be the inclusion map.
Let $f_\beta \in \{f_\alpha\}_{\alpha \in J}$. Then

$$\pi_\beta \circ i : Y \to I_\beta \quad \text{is the desired extention.}$$

Thm 38.4 Let X be comp. reg. Let Y be a compactification of X satisfying the extention property of Thm 38.2. Given any cont. map $f: X \to C$ of X into a compact H. sp. C the map f extends uniquely to a cont. map $g: Y \to C$.

Pf C is comp. reg. Thus C can be embedded in $[0,1]^J$ for some J. We shall regard C as a subsp. of $[0,1]^J$. Then ~~each~~ we can regard f as a map into $[0,1]^J$. We can then write $f(x) = (f_\alpha(x))_{\alpha \in J}$. where $f_\alpha : X \to [0,1]$. So each f_α is a bounded cont. func. $X \to \mathbb{R}$.

Therefore each f_α can be extented to a cont. func. $g_\alpha : Y \to \mathbb{R}$. Put these together to create $g = (g_\alpha)_{\alpha \in J}$ and a cont. map $Y \to \mathbb{R}^J$.

We claim the image of g is still in C.

$$g(Y) = g(\bar{X}) \subset \overline{g(X)} = \overline{f(X)} \subset \bar{C} = C.$$

See Thm 18.1 pg 104. $\blacksquare$

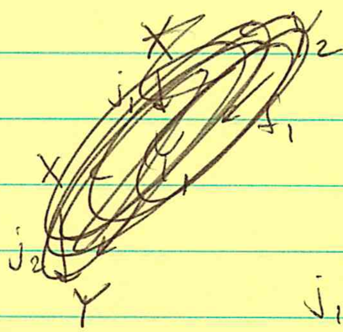
Thm 38.5 Let X be a comp. reg. sp. If Y_1, Y_2 are two compactions satisfying the ext. property of Thm 38.2 then Y_1 and Y_2 are equivalent.

Pf Let $j_i : X \to Y_i$ be inclusion maps.

We can extend $j_2 : X \to Y_2$ to a map $f_2 : Y_1 \to Y_2$.

We can extend $j_1 : X \to Y_1$ to a map $f_1 : Y_2 \to Y_1$.

We claim f_1 and f_2 are homeo's and that they ~~for~~ are the id on X. The latter is obvious.



$$\begin{tikzcd} X \arrow[d, "j_2"'] & \subset Y_1 \arrow[dl, "f_2"] \\ Y_2 & \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[d, "j_1"'] & \subset Y_2 \arrow[dl, "f_1"] \\ Y_1 & \end{tikzcd}$$

$f_1 \circ f_2 : Y_1 \to Y_1$ is the id on X. Therefore

$f_1 \circ f_2$ is a cont. ext of $id : X \to X \subset Y_1$.
But id on Y_1 is also a cont. ext. of id on X.
By uniqueness of extentions (Lemma 38.3) $f_1 \circ f_2 = id_{Y_1}$.
Likewise $f_2 \circ f_1 = id_{Y_2}$.

#4 Let K be any compactification of X. Then ~~any~~ $\exists\ f: \beta X \xrightarrow[\text{cont}]{} K$ that is the identity on X and any cont bounded map $g: X \to \mathbb{R}$ the extention $g': \beta X \to \mathbb{R}$ and $g'': K \to \mathbb{R}$ are related by $g' = g'' \circ f$.

Pf The first statement comes from Thm 38.4
The second from the uniqueness of extentions.

<u>Another point of view.</u>

Let $C^*(X)$ = all bdd cont real valued functions on X. It is a ring.

Recall ring ideals: Let R be a commutative ring with a unit. An ideal $I \subset R$ is an additive subgroup that is closed under multiplication by any ring element.

Ex In $\mathbb{Z}$: multiples of 2, 3, 6 etc
In $\mathbb{R}[x]$: $((x-a))$. $(x-a)(x-b)$
In $C^*(X)$: $\{f(x) \mid f(a)=0\}$ or $\{f(x) \mid f(a)=f(b)=0\}$.

An ideal is maximal if $I \neq R$ and $I \subset J \Rightarrow I = J$.

Let $M(R)$ be the set of maximal ideals of R.

We will put a topology on $M(R)$ as follows.

Let $\mathcal{H}$ be any subset of $\mathcal{M}(R)$.

The kernel of $\mathcal{H}$ is $\bigcap \mathcal{H}$ which is an ideal.

The hull of an ideal I is $\{M \in \mathcal{M}(R) \mid I \subset M\}$.

Ex Let $\mathcal{H} = \{(2), (3)\}$. Kernel $\mathcal{H} = (6)$, ~~the kernel~~ ing.
Hull $(6) = \mathcal{H}$.

Define closure$(\mathcal{H})$ to be the hull of the kernel of $\mathcal{H}$.

Let $C \subset \mathcal{M}(R)$ be closed iff closure$(C) = C$.

This determines a topology on $\mathcal{M}(R)$.

See Theorem 3.7 of Willard's "General Topology".

Ex Consider $\mathcal{M}(C(\mathbb{R}))$. Let $\mathcal{H} = \{\{f \mid f(p)=0\} \mid p \in (2,4)\}$.

Kernel $\mathcal{H} = \{f \mid f(p) = 0, \forall p \in (2,4)\}$.

hull(Kernel $\mathcal{H}$) $= \{\{f \mid f(p)=0\} \mid p \in [2,4]\}$.

Thm βX is homeomorphic to $\mathcal{M}(C^*(X))$.

See: "The Stone-Čech Compactification" by Russell Walker.

#10 Think of β as an opperation that sends c. reg. sp.s to compact. H. sps. $\beta : X \to \beta X$.

Let $f: X \to Y$ be cont. X, Y, c. reg.
$\exists!$ cont. ext. of f to $g: \beta X \to Y$.
Let $i: Y \hookrightarrow \beta Y$ be inclusion.
Then $i \circ g: \beta X \to \beta Y$. Call this map βf.

It is easy to show that $\beta\, id_X = id_{\beta X}$
and that if
$f: X \to Y$ and $g: Y \to Z$ then
$\beta(g \circ f) = \beta g \circ \beta f$.

This mean that β is a <u>functor</u> from the <u>catagory</u> of (c. reg sp, cont func's) $\to$ (cpt H sp., cont. fncs.)