Section 54 $\quad \pi_1(S^1) \cong \mathbb{Z}$

__Def__ Let $p: E \to B$ be a covering map (actually we only need p to be a function). Let $f: X \to B$ be cont. A function $\tilde{f}: X \to E$ s.t. $p \circ \tilde{f} = f$ is called a __lifting__ of f.

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"] & B \end{tikzcd}$$

We will be mainly interested in the case where p is a covering map and $f: I \to B$ is a path, ie, we will study __path liftings.__

__Lemma (54.1)__ Let $p: E \to B$ be a covering map. Let $b_0 \in B$ and fix $e_0 \in p^{-1}(b_0)$. Let $f: I \to B$ be a path with $f(0) = b_0$. Then f has a lifting, $\tilde{f}$, and if we require $\tilde{f}(0) = e_0$, then $\tilde{f}$ is unique.

The proof uses the Lebesgue Number Lemma: Let $\mathcal{A}$ be an open cover of a compact metric space (X, d). $\exists\, \delta > 0$ s.t. $\forall\, B \subset X$ with $\operatorname{diam}(B) < \delta$, $\exists\, A \in \mathcal{A}$ s.t. $B \subset A$. (See page 175.)

Pf of 54.1

Let $\mathcal{U}$ be a covering of B by open sets each of which is evenly covered by p. Then $\{f^{-1}(U) \mid U \in \mathcal{U}\}$ is an open covering of $[0,1]$.

Let $\delta > 0$ be a Lebesgue number for $\{f^{-1}(U) \mid U \in \mathcal{U}\}$. Let $0 = s_0 < s_1 < s_2 < s_3 < \cdots < s_n = 1$ be a partition of $[0,1]$ s.t. $|s_{i+1} - s_i| < \delta$ for $i = 0, 1, 2, \ldots, n-1$. Then for $i = 0, 1, 2, \ldots, n-1$ we know $f([s_i, s_{i+1}]) \subset U$ for some $U \in \mathcal{U}$.

We shall construct $\tilde{f}: I \to E$ is steps and then check uniqueness.

Define $\tilde{f}(0) = e_0$. Let $s \in [0, s_1]$. The set $f([0, s_1])$ lies in some $U \in \mathcal{U}$ which is evenly covered by p. Let $\{V_\alpha\}$ be a partition of $p^{-1}(U)$ where each V_α is homeomorphic to U with $p|V_\alpha$ the homeomorphism. Then e_0 is in one of these, say $e_0 \in V_{\alpha_0}$. Now we can define

$$\tilde{f}(s) = (p|V_{\alpha_0})^{-1}(f(s)) \quad \text{for } s \in [0, s_1].$$

Next let $s \in [s_1, s_2]$. The set $f([s_1, s_2])$ lies in some $U \in \mathcal{U}$. Again let $\{V_\alpha\}$ be a partition of $p^{-1}(U)$ where $p|V_\alpha : V_\alpha \to U$ is a homeo. Let $\tilde{f}(s_1) \in V_{\alpha'}$. Define

$$\tilde{f}(s) = (p|V_{\alpha'})^{-1}(f(s)) \quad \text{for } s \in [s_1, s_2].$$

We continue in the way. After a finite number of steps we will have defined $\tilde{f}$ on all of $[0,1]$. The Pasting lemma shows $\tilde{f}$ is cont.

$p \circ \tilde{f}(s) = p\left(p^{-1}|_{V_{\alpha'}}\right)(f(s)) = f(s)$ for some $V_{\alpha'}$.

The prove of uniqueness is also done step-by-step in a similar manner. See textbook. $\boxtimes$

<u>Lem 54.2</u> Let $p: E \to B$ be a covering map with $b_0 = p(e_0)$. Let $H: I^2 \to B$ be cont. with $H(0,0) = b_0$. There is a unique lifting of H to a cont. map $\tilde{H}: I^2 \to E$ s.t. $\tilde{H}(0,0) = e_0$. If H is a path homotopy, so is $\tilde{H}$.

<u>Pf</u> Similar to 54.1 but more tedious. See textbook.

<u>Thm 54.3</u> Let $p: E \to B$, $p(e_0) = b_0$, be a covering map. Let f and g be paths in B from b_0 to b_1. Let $\tilde{f}$ and $\tilde{g}$ be their lifts that begin at e_0. If f and g are path homotopic, then $\tilde{f}(1) = \tilde{g}(1)$ and $\tilde{f}$ and $\tilde{g}$ are path homotopic in E.

<u>Pf</u> Apply the last two lemmas. See textbook. $\square$

Def Consider a covering map $p: E \to B$ with $p(e_0) = b_0$. We define a function $\Phi: \pi_1(B, b_0) \to p^{-1}(b_0)$ as follows. Let $[f] \in \pi_1(B, b_0)$. Let $\tilde{f}$ be the lift of f with $\tilde{f}(0) = e_0$. Then $p(\tilde{f}(1)) = \cancel{f}(1) = b_0$. Let $\Phi([f]) = \tilde{f}(1) \in p^{-1}(b_0)$.

Thm 54.3 shows Φ is well defined.

Lem 54.4 Let $p: E \to B$, $p(e_0) = b_0$, be a covering map. If E is path connected, then Φ is surjective (onto). If E is simply connected, then Φ is a bijection.

Pf Let E be path conn'd. Let $e_1 \in p^{-1}(b_0)$. Let $\lambda: I \to E$ be a path from e_0 to e_1. Let $f = p \circ \lambda$. Then $[f] \in \pi_1(X, b_0)$ since f is a loop based at b_0. Now $\Phi([f]) = \tilde{f}(1)$. But λ is also a lift of f that starts at e_0. Thus, $\tilde{f}(1) = \lambda(1) = e_1$.

Let E be simply conn'd. Let $[f], [g] \in \pi_1(B, b_0)$ with $\Phi([f]) = \Phi([g])$. We claim $f \simeq_p g$, and hence $[f] = [g]$. Let $\tilde{f}$ and $\tilde{g}$ be the lifting of f and g resp with $\tilde{f}(0) = \tilde{g}(0) = e_0$. Then $\tilde{f}(1) = \tilde{g}(1)$.

Since E is simply conn'd, by Lemma 52.3, we know $\tilde{f} \simeq_p \tilde{g}$. Let $\tilde{H}$ be a path homotopy from $\tilde{f}$ to $\tilde{g}$. Then you can check $p \circ \tilde{H}$ is a path homotopy from f to g. Hence $[f] = [g]$. $\square$

Thm 54.5 $\quad \pi_1(S^1, (1,0)) \cong \mathbb{Z}$.

Outline of Proof Let $S^1 = \{(x,y) \mid x^2+y^2=1\}$, $b=(1,0)$.
Let $p : \mathbb{R} \to S^1$ be given by

$$p(t) = \langle \cos(2\pi t), \sin(2\pi t) \rangle.$$

Then $p^{-1}(b) = \mathbb{Z}$. Let $e_0 = 0 \in \mathbb{Z}$.

Since $\mathbb{R}$ is simply connected $\Phi : \pi_1(S, (1,0)) \to \mathbb{Z}$ is a bijection. We will check that Φ is a homorphism.

Let $[f], [g] \in \pi_1(S^1, b)$. Suppose

$$\Phi([f]) = \tilde{f}(1) = m,$$
$$\Phi([g]) = \hat{g}(1) = n.$$

Let $\hat{g} = \tilde{g} + m$. ($\hat{g} : I \to \mathbb{R}$ is the unique lift of g with $\hat{g}(0) = m$.)
Now $\tilde{f} * \hat{g}$ is defined and you can check

$$\widetilde{f*g} = \tilde{f} * \hat{g} \quad \text{(The unique lift of } f*g \text{ starting at } 0.)$$

Then
$$\Phi([f]*[g]) = \Phi([f*g]) = (\tilde{f} * \hat{g})(1) = \hat{g}(1) = n+m.$$

Thus, $\pi(S^1, (1,0)) \cong \mathbb{Z}$. $\square$

<u>Note</u> Read Thm 54.6 on your own. We will come back to it later if time permits.