§58 Homotopy Type

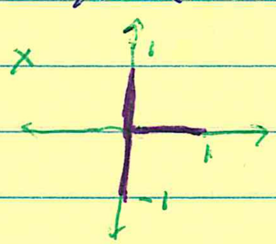
Def Two spaces X and Y have the same **homotopy type** if $\exists$ continuous maps $f: X \to Y$ and $g: Y \to X$ s.t.

$$f \circ g \simeq id_Y \text{ and } g \circ f \simeq id_X.$$

We may also say two such spaces are **homotopic**. The maps f and g are **homotopy equivalences** and they are **homotopy inverses** of each other.

Fact Homotopy equivalence is an equivalence relation on top. sp's that is strictly courser than top eq; obviously homeomorphic spaces are homotopic.

Ex Let $X = \{0\} \times [-1,1] \cup [0,1] \times \{0\}$ and $Y = \{0\} \times [-1,1]$.



X and Y are not homeo. (we cut pts), but we show that they are homotopic.

Let $g: Y \to X$ be inclusion and let $f: X \to Y$ be

$$f(x,y) = \begin{cases} (0,y) & \text{if } x = 0, \\ (0,0) & \text{if } x \neq 0. \end{cases}$$

Clearly $f \circ g = id_Y$, so we only need to show $g \circ f \simeq id_X$.

We define a homotopy $F : X \times I \to X$ by

$$F((x,y),t) = \begin{cases} (0,y) & \text{if } x = 0, \\ (tx, 0) & \text{if } x \neq 0. \end{cases}$$

(we do not really need branched function notation here.)

Then,

$$F((x,y),1) = id_X(x,y) \text{ and}$$

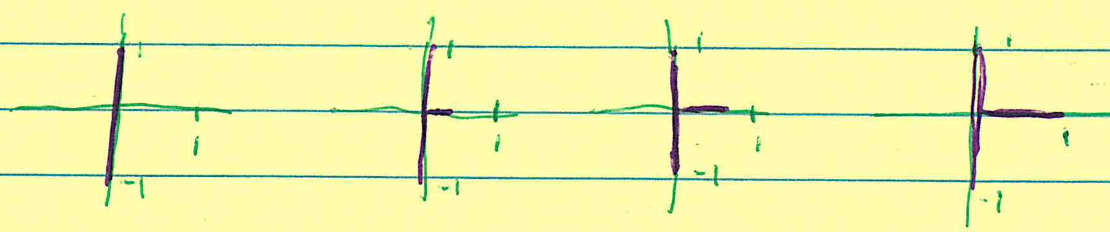
$$F((x,y),0) = \begin{cases} (0,y) & \text{if } x = 0, \\ (0,0) & \text{if } x \neq 0, \end{cases} = f(x,y).$$

But $f(x,y) = g(f(x,y))$, thus

$$F((x,y),0) = (g \circ f)(x,y).$$

Hence $g \circ f \simeq id_X$.

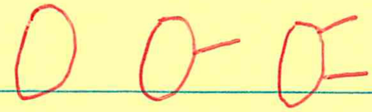
Here we show some stages of the homotopy.



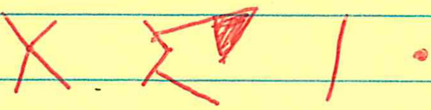
$t = 0$ $\quad$ $t = \frac{1}{3}$ $\quad$ $t = \frac{2}{3}$ $\quad$ $t = 1$

Here shows the image of X under F for fixed values of t.

re Ex's



are homotopic but not homeo.



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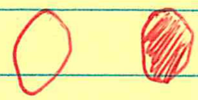
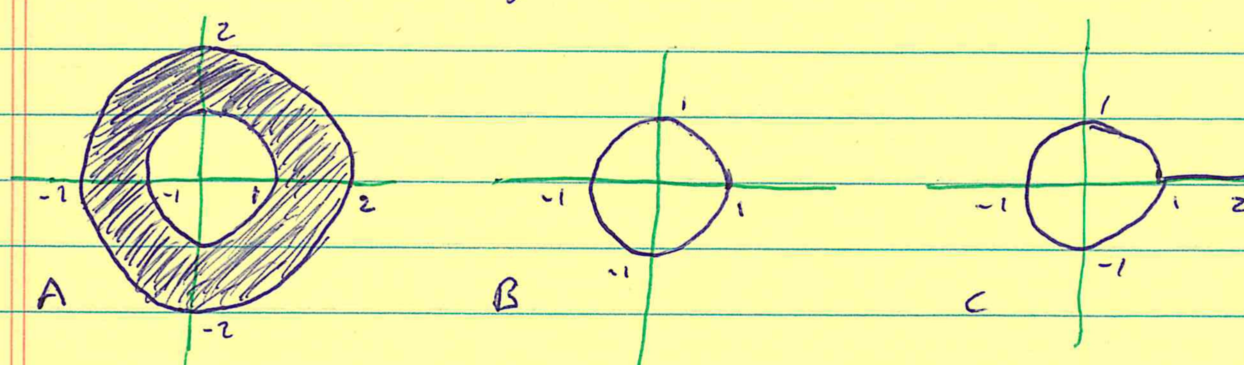


are not homotopic, as we shall see later.

Def Let $A \subset X$. Let $r: X \to A$ be a retraction. If r is homotopic to id_X, then r is a deformation retraction and A is a deformation retract of X.

This is a special type of homotopy equivalence. The first example we gave was a def. retraction.

Ex



There is a def. ret. from the annulus in A to the circle in B and from the set in C to the circle in B. Hence the sets in A and C are homotopic. In fact there is a def. ret. from A to C. Try to visualize it.

Thm (58.7) Let $f: X \to Y$ be a homotopy eq. Then

$$\pi_1(X, x_0) \cong \pi_1(Y, f(x_0)).$$

Ex's Thus π_1 of the annulus and  are isomorphic to $\mathbb{Z}$.

Since $\pi_1(S^1)$ and $\pi_1([0,1])$ are not isomorphic, S^1 and $[0,1]$ are not homotopic.

The proof of Thm 58.7 uses the very painful lemma 58.4. We cover it next.

Thm 58.4 Let h and k be cont. maps from X to Y.
Let $x_0 \in X$, $y_0 = h(x_0)$ and $y_1 = k(x_0)$.
If h and k are homotopic maps then $\exists$ a path α in Y from y_0 to y_1 s.t. $k_* = \hat{\alpha} \circ h_*$.

Pf Recall from pg 331 that $\hat{\alpha} : \pi_1(Y, y_0) \to \pi_1(Y, y_1)$ is given by $\hat{\alpha}([f]) = [\bar{\alpha}] * [f] * [\alpha]$.

Let $H : X \times I \to Y$ be a homotopy between h and k.
Define $\alpha : I \to Y$ by

$$\alpha(t) = H(x_0, t) \quad , \ t \in [0,1].$$

Notice $\alpha(0) = H(x_0, 0) = h(x_0) = y_0$ and
$\alpha(1) = H(x_0, 1) = k(x_0) = y_1$.
Thus α is a path from y_0 to y_1 in Y.

For $[f] \in \pi_1(X, x_0)$ we need to show that

$$k_*([f]) = (\hat{\alpha} \circ h_*)([f]).$$

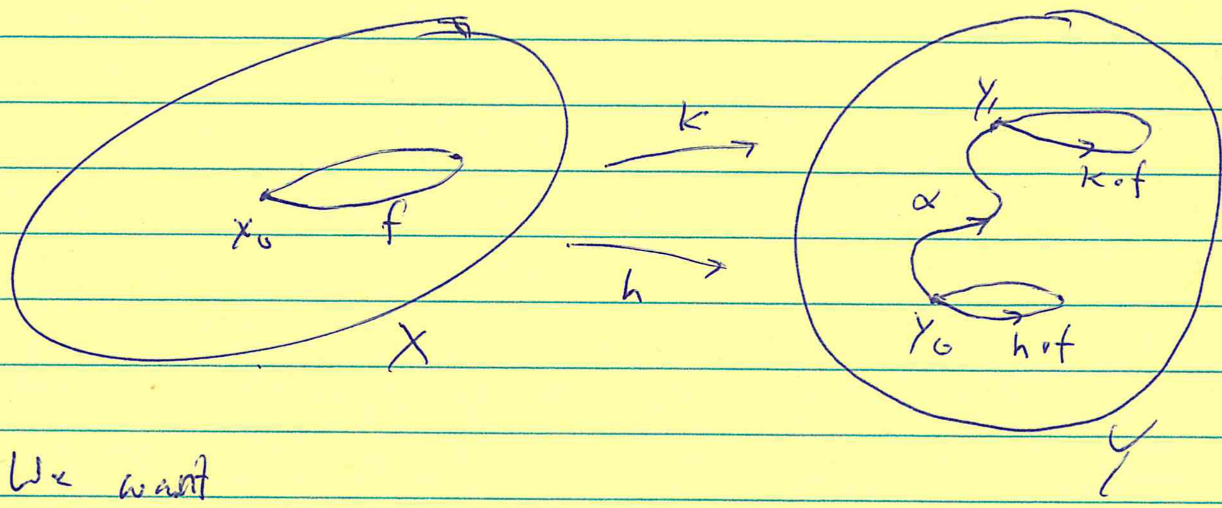
$\iff \quad [k \circ f] = [\bar{\alpha}] * [h \circ f] * [\alpha]$

$\iff \quad [\alpha] * [k \circ f] = [h \circ f] * [\alpha] \qquad$ (not in gp)

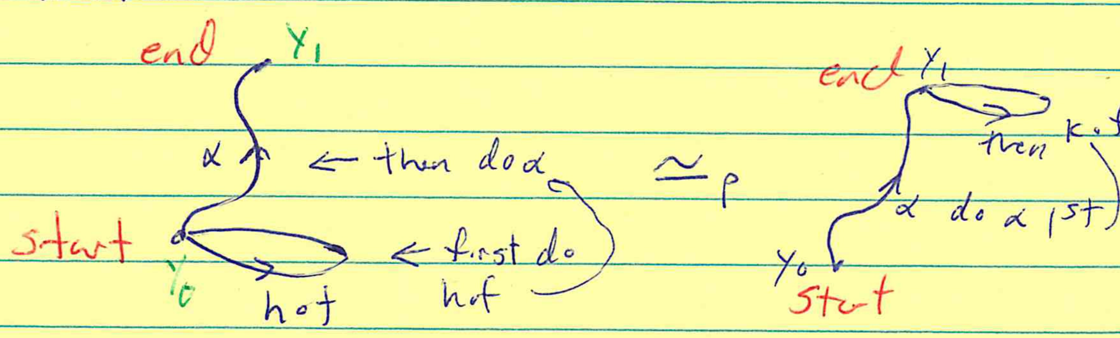
$\iff \quad \alpha * (k \circ f) \simeq_p (h \circ f) * \alpha$

We illustrate the homotopy on the next page, and then work out the formal proof.

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[r, "h_*"] \arrow[dr, "k_*"'] & \pi_1(Y, y_0) \arrow[d, "\hat{\alpha}"] \\ & \pi_1(Y, y_1) \end{tikzcd}$$



We want

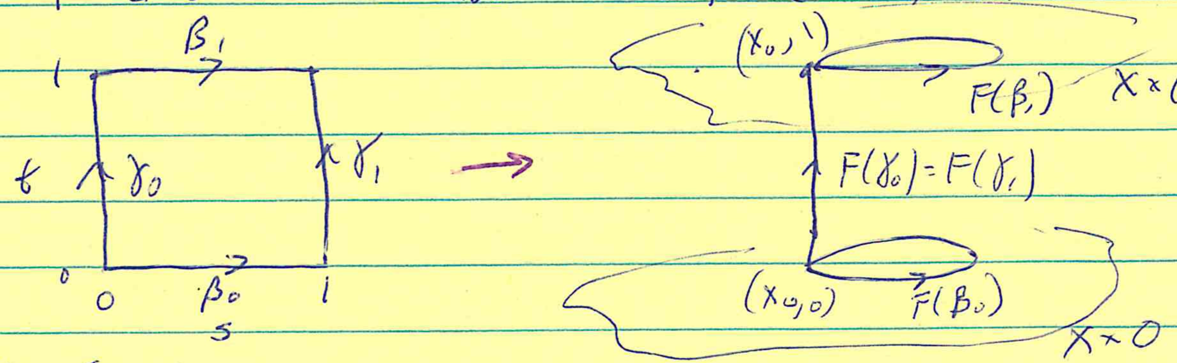


This path homotopy, $P: I \times I \to Y$, will have

$$P(s,0) = ((h \circ f) * \alpha)(s)$$
$$P(s,1) = (\alpha * (k \circ f))(s)$$
$$P(0,t) = y_0$$
$$P(1,t) = y_1.$$

We will construct it in stages.

Step 1 Let $F : I \times I \to X \times I$ be $F(s,t) = (f(s), t)$.



$\beta_0(s) = (s, 0)$, $s \in I$

$\beta_1(s) = (s, 1)$ $s \in I$

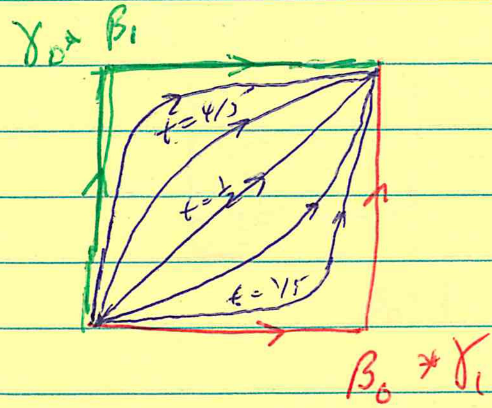
$\gamma_0(t) = (0, t)$ $t \in I$

$\gamma_1(t) = (1, t)$ $t \in I$

Let $c(t) = (x_0, t)$ in $X \times I$.
(We be 'pre-image' of α)

Step 2 We claim $F(\beta_0) * c \simeq_p c * F(\beta_1)$ in $X \times I$.
This is the same as $F(\beta_0) * F(\gamma_1) \simeq_p F(\gamma_0) * F(\beta_1)$.

To find such a homotopy we let $G : I \times I \to I \times I$ be a path homotopy from $\beta_0 * \gamma_1$ to $\gamma_0 * \beta_1$.



$G(s, 0) = \beta_0 * \gamma_1(s)$.

$G(s, 1) = \gamma_0 * \beta_1(s)$.

$G(0, t) = (0, 0)$.

$G(1, t) = (1, 1)$.

Now let $Q = F \circ G : I \times I \xrightarrow{G} I \times I \xrightarrow{F} X \times I$.

Then $Q(s,0) = F(G(s,0)) = F(\beta_0 * \gamma_1(s))$

$$= \begin{cases} F(\beta_0(2s) & s \in [0,\frac{1}{2}] \\ F(\gamma_1(2s-1)) & s \in [\frac{1}{2},1] \end{cases} = \begin{cases} (f(2s),0) & s \in [0,\frac{1}{2}] \\ (x_0, 2s-1) & s \in [\frac{1}{2},1] \end{cases}$$

$$= (f(s),0) * c = F(\beta_0) * c.$$

And $Q(s,1) = F(G(s,1)) = F(\gamma_0 * \beta_1(s)) =$

$$\begin{cases} F(\gamma_0(2s) & s \in [0,\frac{1}{2}] \\ F(\beta_1(2s-1)) & s \in [\frac{1}{2},1] \end{cases} = \begin{cases} (x_0, 2s) & s \in [0,\frac{1}{2}] \\ (f(2s-1),1) & s \in [\frac{1}{2},1] \end{cases}$$

$$= c * (f(s),1) = c * F(\beta_1).$$

<u>Step 3</u> Now we push through to Y. Let

$$P = H \circ Q = H \circ F \circ G : I \times I \xrightarrow{G} I \times I \xrightarrow{F} X \times I \xrightarrow{H} Y$$

We just use the definitions to check that P

$$P(s,0) = H((f(s),0) * c) = (h \circ f) * \alpha \qquad (\alpha \text{ is the image of } c)$$

$$P(s,1) = H(c * (f(s),1)] = \alpha * (k \circ f).$$

$$P(0,t) = H((x_0,0)) = h(x_0) = y_0.$$

$$P(1,t) = H((x_0,1)) = k(x_0) = y_1.$$

Therefore $\alpha * (k \circ f) \simeq_p (h \circ f) * \alpha$,

which implies $[k \circ f] = [\bar{\alpha}] * [h \circ f] * [\alpha]$,

that is $k_*([f]) = \hat{\alpha}(h_*([f]))$. $\blacksquare$

Next we prove Thm 58.7

Pf of Thm 58.7 Let $f: X \to Y$ be a homotopy eq. Let $x_0 \in X$, $y_0 = f(x_0) \in Y$, We will show

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

is a isomorphism.

Let $g: Y \to X$ be a homotopy inverse of f. Let $x_1 = g(y_0) \in X$. (x_1 need not equal x_0.) Consider,

$$(X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{g} (X, x_1) \xrightarrow{f'} (Y, y_1)$$

where $f'(x) = f(x)$, $y_1 = f'(x_1)$; we use f' instead of f because the base pt is not the same and so f_* and f'_* are different; they have different domains.

Now $g \circ f : (X, x_0) \to (X, x_1)$ and $g \circ f \simeq id_X$. By Lemma 58.4

$$(g \circ f)_* = \hat{\alpha} \circ (id_X)_* = \hat{\alpha}.$$

for some path $\alpha: I \to X$ from x_0 to x_1. Now $\hat{\alpha}$ is an iso. by Lemma 52.1. Thus $(g \circ f)_* : \pi_1(X, x_0) \to \pi_1(X, x_1)$ is an iso.

Similarly, you can show that $(f' \circ g)_*: \pi_1(Y, y_0) \to \pi_1(Y, y_1)$ is an iso.

Now, since $(g \circ f)_* = g_* \circ f_*$ is an iso f_* is one-to-one and g_* is onto.

And, since $(f' \circ g)_* = f'_* \circ g_*$ is an iso, g_* is one-to-one and f'_* is onto.

Thus g_* is one-to-one and onto and hence an iso.

Finally, to show f_* is an iso use

$$g_* \circ f_* = \hat{\alpha} \Rightarrow g_*^{-1} \circ \hat{\alpha} = f_*.$$

Thus f_* is a composition of isomorphisms and hence f_* is an iso. $\square$