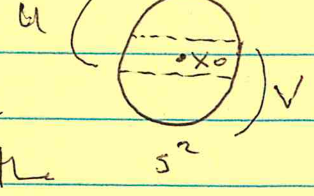
§59 S^n

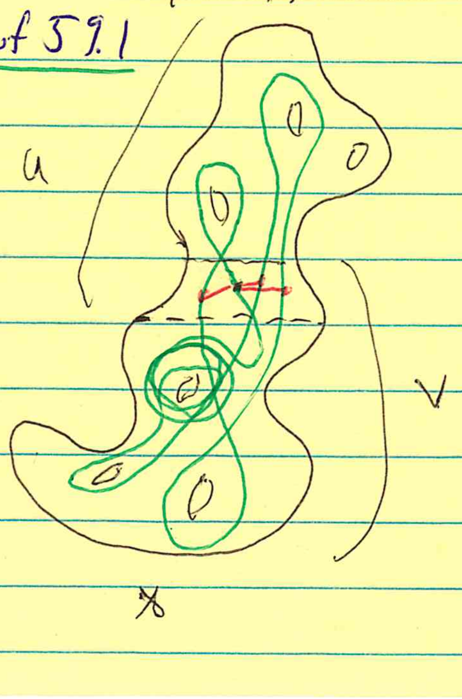
Thm 59.1 Suppose $X = U \cup V$ where U and V are open, $U \cap V$ is path connected and let $x_0 \in U \cap V$. Let $i : U \hookrightarrow X$ and $j : V \hookrightarrow X$ be inclusion maps. Then $\pi_1(X, x_0)$ is generated by the images of i_* and j_*.

Application $\pi_1(S^2, -)$ is trivial. Proof:



Let U be the subset of S^2 whose z coord. is $> -\frac{1}{3}$ and let V be the subset of S^2 with $z < \frac{1}{3}$. Now U and V are homeo to open disks. Let $x_0 \in U \cap V$, say $(1, 0, 0)$. Then $\pi_1(U, x_0)$ and $\pi_1(V, x_0)$ are trivial groups. The images of $i_* : \pi_1(U, x_0) \to \pi_1(S^2, x_0)$ and $j_* : \pi_1(V, x_0) \to \pi_1(S^2, x_0)$ can only be the identity element of $\pi_1(S^2, x_0)$. By Thm 59.1 $\pi_1(S^2, x_0)$ is generated by its identity element and hence it too is just the trivial group. $\square$ [This works for any S^n, $n \geq 2$.]

Idea of Pf of 59.1

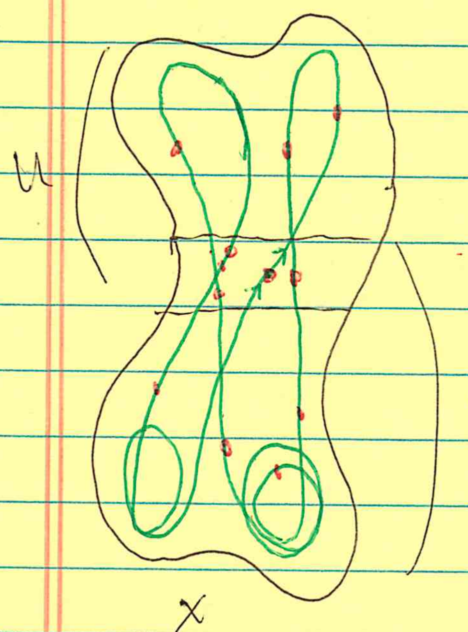


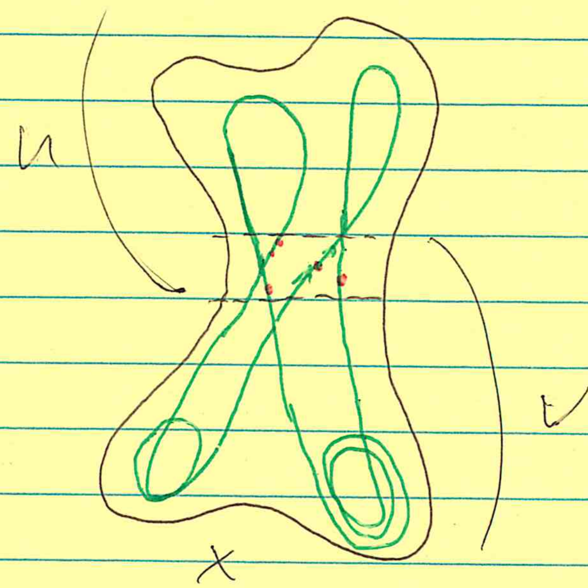
Any loop in X based at x_0 factors into loops only in U or only in V based at x_0.

Proof Let $f: I \to X$ be a loop based at x_0. By the Lebesgue number lemma (pg 175) $\exists$ a ~~partition~~ partition of I, $0 = a_0 < a_1 < a_2 < \cdots < a_n = 1$, s.t.

$$f([a_i, a_{i+1}]) \subset U \text{ or } V, \quad i = 0, \dots, n-1.$$

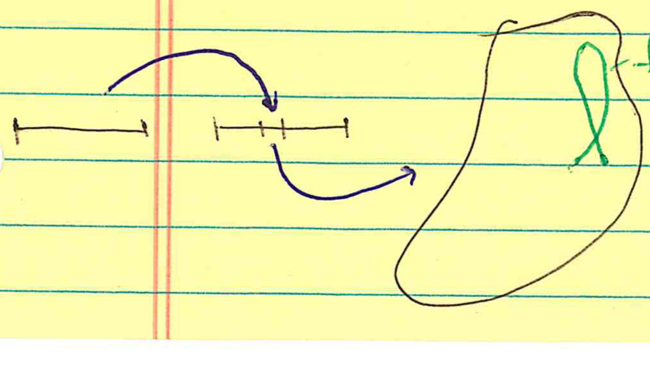
We remove from $\{a_i\}_{i=0}^{n}$ any a_i's s.t. $f(a_i) \notin U \cap V$,

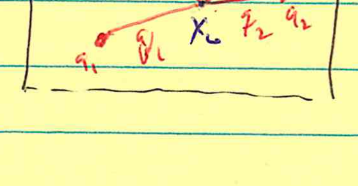




and renumber the remaining $\{a_i\}_{i=0}^{k}$, $k \leq n$, and still have $f([a_i, a_{i+1}]) \subset U$ or V $i = 0, \dots, k$.

Let q_i be a path in $U \cap V$ from x_0 to a_i for $i = 1, \dots, k-1$. Let $f_i : I \to X$ be a reparameterization of $f|[a_i, a_{i+1}]$.





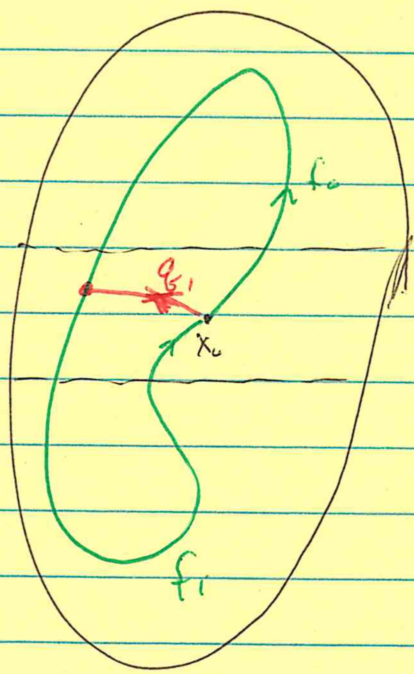
We can write $f \simeq_p f_0 * f_1 * f_2 * \cdots * f_k$. However, the f_i's are not loop in general. But we can write

$$f \simeq f_0 * \bar{q}_1 * q_1 * f_1 * \bar{q}_2 * q_2 * f_2 * \bar{q}_3 * \cdots * \bar{q}_{k-1} * q_{k-1} * f_k .$$

Now

$$[f] = [f_0 * \bar{q}_1] * [q_1 * f_1 * \bar{q}_2] * [q_2 * f_2 * \bar{q}_3] * \cdots * [q_{k-1} * f_k]$$

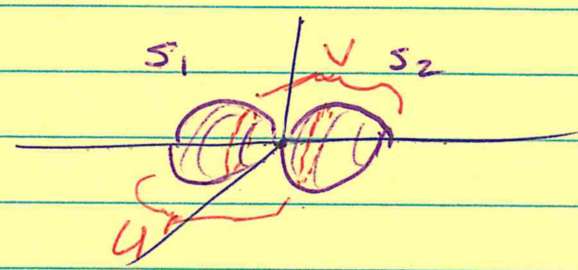
and each term inside a pair of brakets is a loop based a x_0 and it is wholely in U or V. Thus $[f]$ has been factored into members of $\pi_1(U, x_0)$ and $\pi_1(V, x_0)$. ■



#1 Let X be the union of two copies of S^2 with a single pt in common. (This is called a <u>wedge product</u> and is denoted $X = S^2 \vee S^2$.) What is $\pi_1(X)$?

Solution Let $S_1 = \{(x,y,z) \in \mathbb{R}^3 \mid (x+1)^2 + y^2 + z^2 = 1\}$ and let $S_2 = \{(x,y,z) \in \mathbb{R}^3 \mid (x-1)^2 + y^2 + z^2 = 1\}$. Then X is homeo. to $S_1 \cup S_2$. Then $0 = (0,0,0)$ is the only element of $S_1 \cap S_2$. But it is not open. Let

$$U = \{(x,y,z) \in S_1 \cup S_2 \mid x < \tfrac{1}{3}\} \text{ and}$$
$$V = \{(x,y,z) \in S_1 \cup S_2 \mid x > -\tfrac{1}{3}\}.$$



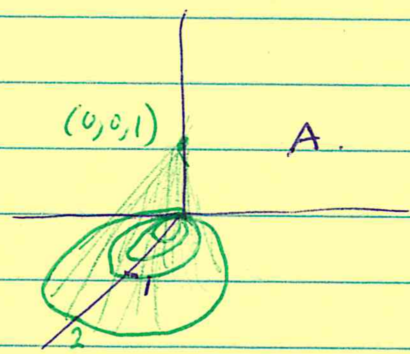
Now U, V and $U \cap V$ are open, and $0 \in U \cap V$. It is easy to show $U \cap V$ is path connected. Thus we can apply Thm 59.1. You can show that $\pi_1(U, 0)$ and $\pi_1(V, 0)$ are trivial by showing that S_1 is a deformation retract of U and S_2 is a def. ret. of V. Hence $\pi_1(S_1 \cup S_2, 0)$ is trivial.

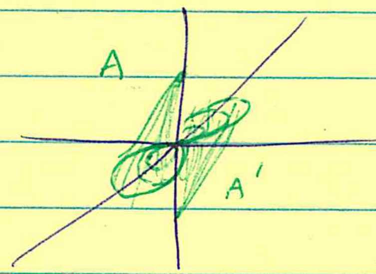
It is easy to show $S_1 \cup S_2$ is path conn'd so the result does not depend on the choice of base pt. Thus $\pi_1(S^2 \vee S^2)$ is always trivial. ▣

The text mentions an example of two simply conn'd space having only one pt in common, ~~b those fun~~ but where the fund. gp. of their union is not trivial. It show why the assumption that U and V are open cannot by dropped in Thm 59.1. The example is in Alg. Top. by Spanier, pg 59. It goes like this.

Let $C_n = \{(x, y, 0) \in \mathbb{R}^3 \mid (x - \frac{1}{n})^2 + y^2 = \frac{1}{n^2}\}$.

Let A = the set of pts on any closed line segment from (0, 0, 1) to a pt on any C_n, $n = 1, 2, 3, \ldots$.





Let A' be the reflection of A through the origin:
$A' = \{(x, y, z) \in \mathbb{R}^3 \mid (-x, -y, -z) \in A\}$.

$\exists$ deformation retractions of A to $\{(0,0,1)\}$ and of A' to $\{(0,0,-1)\}$, so both have trivial fundamental gps. Note $A \cap A' = \{(0,0,0)\}$.

Yet, it can be shown that $\pi_1(A \cup A', 0)$ is not trivial, in fact it cannot be finitely generated.