§60 Some Surfaces and Products.

Def On S^2 we identify the antipodal points, $(x,y,z) \sim (-x,-y,-z)$. The quotient space is called the projective plane, P^2.

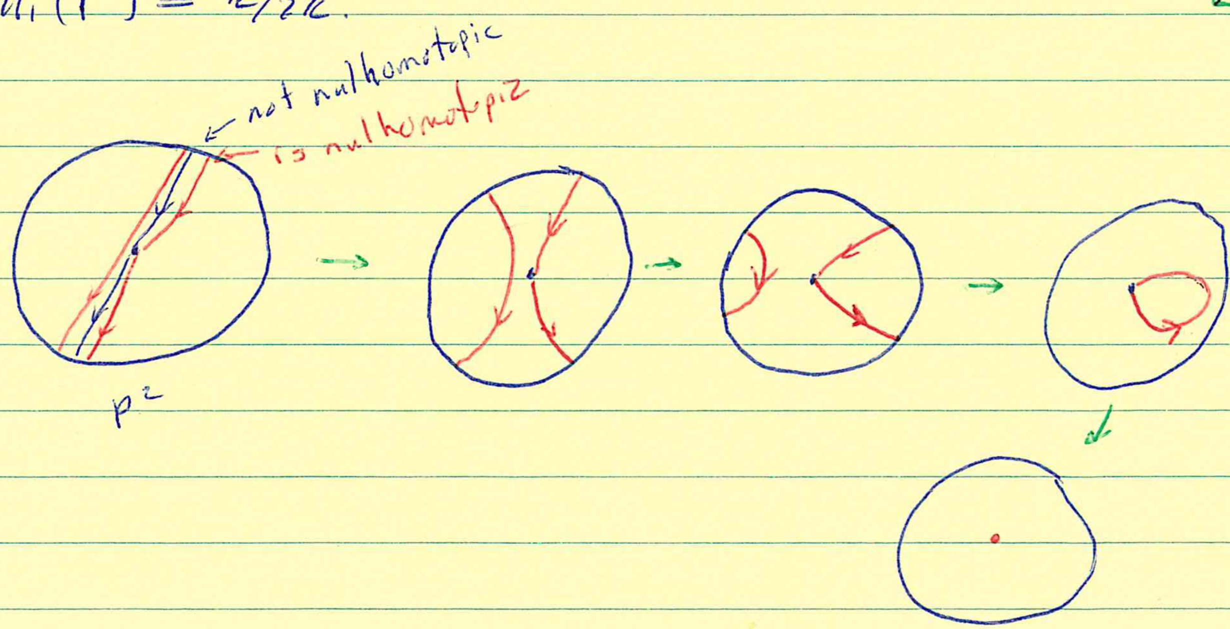
Fact This quotient map $q: S^2 \to P^2$ is in fact a 2-to-1 covering map, and P^2 is a compact 2-manifold. See Thm 60.3 in textbook.

Thm ((Cor. 60.4 in textbook) $\pi_1(P^2, *) \cong \mathbb{Z}/2\mathbb{Z}$.

Pf We recall Thm 54.4: $p: E \to B$ covering map, and $\pi_1(E)$ trivial $\Rightarrow$ the lifting correspondence

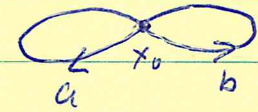
$$\Phi : \pi_1(B, b_0) \to p^{-1}(b_0)$$

is one-to-one. For the map $q: S^2 \to P^2$, $q^{-1}(x)$ always has two members and $\pi_1(S^2)$ is trivial. Therefore $\pi_1(P^2)$ has just two members. Thus $\pi_1(P^2) \cong \mathbb{Z}/2\mathbb{Z}$. $\boxtimes$

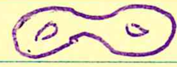


Thm (Lemma 60.5) $\pi_1(\infty)$ is not abelian.

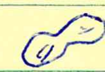
Pf See textbook, but this will be easier to do after we have some group thy. Consider



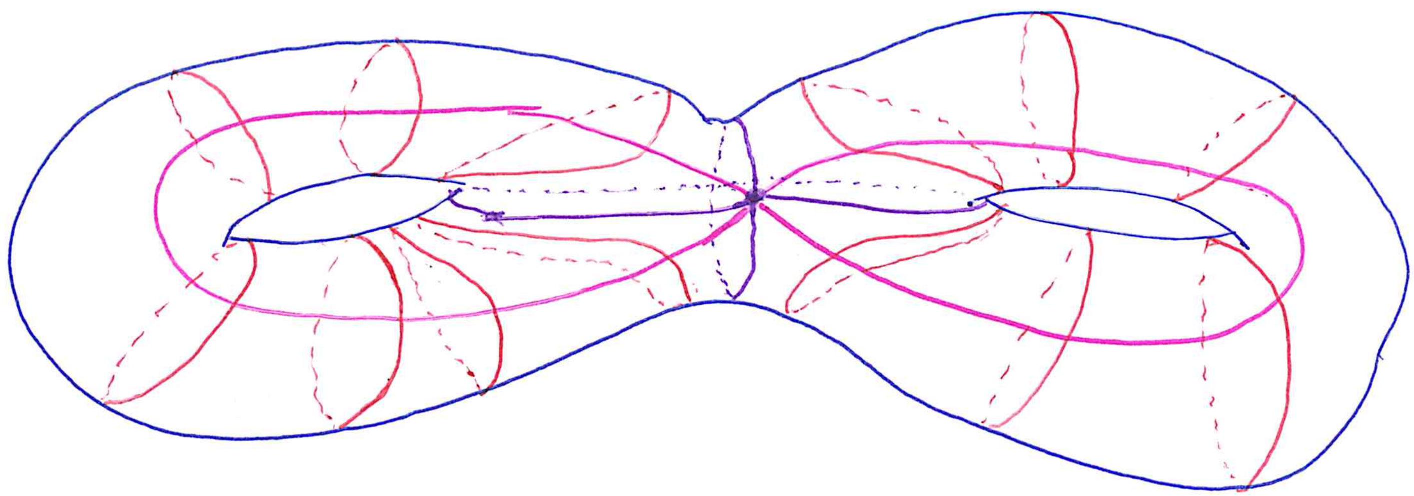
Then Thm 59.1 $\Rightarrow$ $\pi_1(\infty, x_0)$ is generated by $[a]$ and $[b]$. But, there are no "relations," so $\pi_1(\infty)$ is a free group on 2 generators and hence is not abelian. $\boxed{\;}$

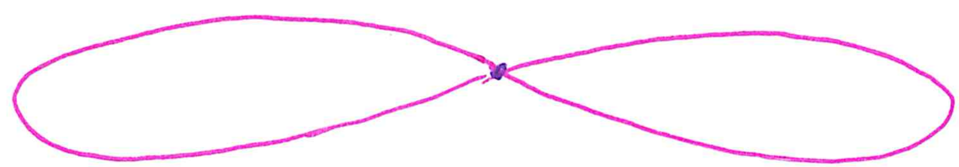
Thm (60.6) $\pi_1($  $)$ is not abelian.

Idea of Pf The figure on the next page shows $\exists$ a retraction from  to ∞. It follows that $\pi_1($  $)$ contains a copy of $\pi_1(\infty)$ as a subgp. Hence $\pi_1($  $)$ is not abelian.



Cor The surfaces S^2, P^2 and $T \# T$ are not homotopically equivalent and hence are not homeomorphic.





Each red circle maps to a pt on one of the two pink loops, the intersection points. The two purple circles (which intersect at two pts) map to the purple point. This is a retraction of $T^2 \# T^2$ to a figure 8.

Products

Thm 60.1 $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \oplus \pi_1(Y, y_0)$.

Pf Let $p: X \times Y \to X$ and $q: X \times Y \to Y$ be projections. Then we have the induced homomorphisms:

$$p_*: \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0), \text{ and}$$
$$q_*: \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(Y, y_0).$$

Define $\Phi: \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \oplus \pi_1(Y, y_0)$ by

$$\Phi([f]) = (p_*([f]), q_*([f])) = ([p \circ f], [q \circ f]).$$

It is easy to check that Φ is a homomorphism. We will show that it is onto and one-to-one.

Onto Let $([g], [h]) \in \pi_1(X, x_0) \oplus \pi_1(Y, y_0)$. Let $f(t) = (g(t), h(t))$. Then $f: [0,1] \to X \times Y$ is a ~~path~~ loop based at (x_0, y_0).

$$\Phi([f]) = ([p \circ f], [q \circ f]) = ([g], [h]). \checkmark$$

One-to-one We will show the kernel of Φ is trivial. Let $[f] \in \pi_1(X \times Y, (x_0, y_0))$ be s.t. $\Phi([f]) = ([e_{x_0}], [e_{y_0}])$. $\exists$ homotopies G and H for $p \circ f \simeq e_{x_0}$ and $q \circ f \simeq e_{y_0}$, resp. Define $F: I \times I \to X \times Y$ by

$$F(s,t) = (G(s,t), H(s,t)).$$

This is a homotopy from f to $e_{(x_0, y_0)} = (e_{x_0}, e_{y_0})$. Check this. Thus $[f]$ is the identity element. $\square$

Ex's $\pi_1(T^2) = \pi_1(S^1 \times S^1) \cong \mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z}^2$.

$\pi_1(T^3) = \pi_1(S^1 \times S^1 \times S^1) \cong \mathbb{Z}^3$

$\pi_1(S^1 \times S^2) \cong \mathbb{Z} \oplus 0 \cong \mathbb{Z}$.

Thus the 3-manifolds T^3, $S^1 \times S^2$ and S^3 are distinct, they are not homeomorphic or homotopic to each other.