Thm (35.1) (Tietze Extension Theorem)

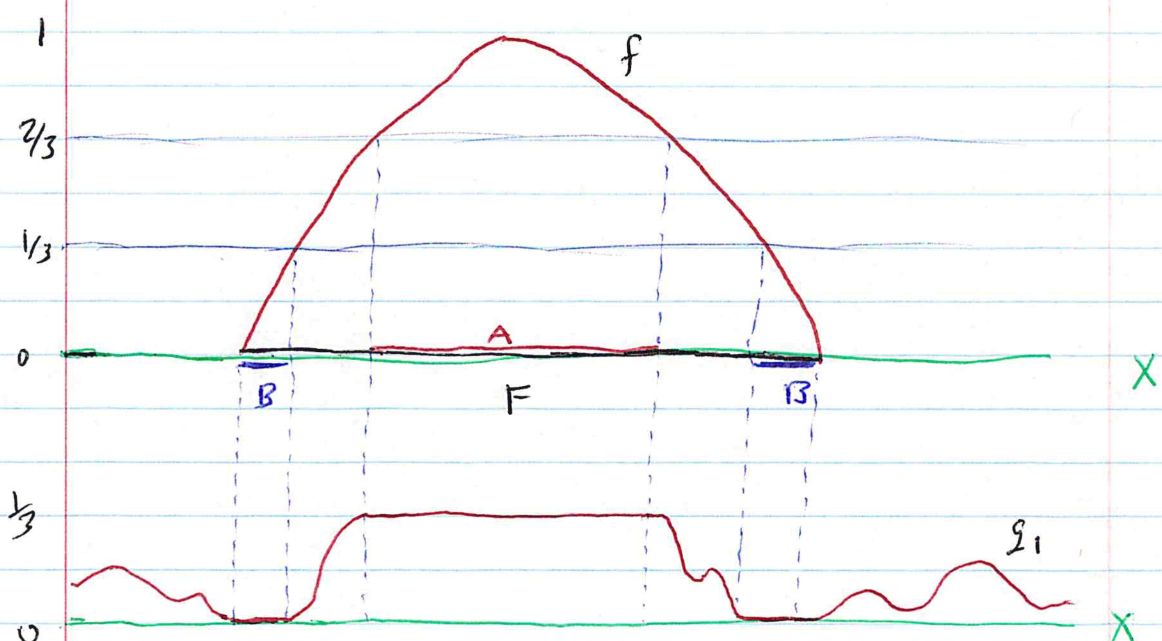
Let X be a normal space, let $F \subset X$ be closed and let $f: F \to [0,1]$ be continuous. Then $\exists$ a cont. func. $g: X \to [0,1]$ s.t. $g|_F = f$.

Rmk This is false if F is not closed. Let $X = \mathbb{R}$ and $F = (0,1) \cup (1,2)$. Then $f(x) = 0$ for $x \in (0,1)$ and 7 for $x \in (1,2)$ is cont. on F but there is no cont. extension of f to $\mathbb{R}$.

Let $X = S^1 = \{(x,y) \mid x^2 + y^2 = 1\} \subset \mathbb{R}^2$. Let $F = S^1 - \{(1,0)\}$. Let $f(x,y) = \theta$. f is cont. on F but does not have a cont. ext. to S^1.

Pf Let $F \subset X$ be closed, X normal, $f: F \to [0,1]$ continuous.

Let $A = \{x \in F \mid f(x) \geq \frac{2}{3}\} = f^{-1}([\frac{2}{3}, 1])$; A is closed.
Let $B = \{x \in F \mid f(x) \leq \frac{1}{3}\} = f^{-1}([0, \frac{1}{3}])$; B is closed.



By Urysohn's Lemma $\exists\, g_1 : X \to [0, \frac{1}{3}]$, cont., s.t.

$$g_1(x) = 0 \quad \text{for } x \in B,$$
$$g_1(x) = 1/3 \quad \text{for } x \in A.$$

Let $f_1 = f - g_1$ on F. Check that $0 \le f_1(x) \le \frac{2}{3}$.

Now repeat, using f_1 in place of f and establishing the existance of $g_2 : X \to [0, \frac{1}{3} \cdot \frac{2}{3}]$ cont. s.t.

$$g_2(x) = 0 \quad \text{when } f_1(x) \le \tfrac{1}{3}\tfrac{2}{3},$$
$$g_2(x) = \tfrac{1}{3}\tfrac{2}{3} \quad \text{when } f_1(x) \ge \tfrac{2}{3}\tfrac{2}{3}.$$

Let $f_2 = f_1 - g_2$ on F. Check that $0 \le f_2(x) \le \frac{2}{3}\frac{2}{3}$.

Continue inductively so that $g_k : X \to [0, \frac{2^{k-1}}{3^k}]$ cont s.t.

$$g_k(x) = 0 \quad \text{when } f_{k-1}(x) \le \frac{2^{k-1}}{3^k},$$
$$g_k(x) = 1 \quad \text{when } f_{k-1}(x) \ge 2^k/3^k.$$

For each k, $0 \le f_k(x) \le (\frac{2}{3})^k$ on F.

Thus, on F, $\sum g_k(x) \to f(x)$ since $f_k(x) \to 0$.

By the Weierstrass M-test $\sum g_k$ converges uniformly on X. Let $g = \sum_{k=1}^{\infty} g_k$. Then g is cont. on X

and $g(x) = f(x)$ for $x \in F$.

$\square$

Note: This is based on "Top and Geom." by Bredon.