Free Groups

We will describe in loose terms the definition the free group on two generators. We start with two symbols, a and b. We specify three other symbols, e, $\bar{a}$ and $\bar{b}$. Consider the set S of all finite words in these symbols.

$$S = \{s_1 s_2 s_3 \cdots s_n \mid s_i = a, b, \bar{a}, \bar{b}, \text{or } e, \text{ and } n > 0\}.$$

We define a binary operation $S \times S \to S$ called *concatenation* as follows. Let $u = s_1 \ldots s_k$ and $v = t_1 \ldots t_m$ be in S. Then

$$uv = s_1 \ldots s_k t_1 \ldots t_m \in S.$$

But S is not a group. We define an equivalence relation on S that will make it into a group. Let $u \sim v$ if u can be transformed into v by a finite sequence of the following moves:

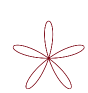
$$\begin{array}{llll} a\bar{a} = e & \bar{a}a = e & b\bar{b} = e & \bar{b}b = e \\ ae = a & ea = a & be = b & eb = b \\ \bar{a}e = \bar{a} & e\bar{a} = \bar{a} & \bar{b}e = \bar{b} & e\bar{b} = \bar{b} \\ ee = e & & & \end{array}$$

Here is an example: $abaee\bar{a}bb\bar{b}ea = abba$. Now we can regrad e as the identity element and $\bar{a}$ and $\bar{b}$ as inverses of a and b, repsecively.

The free group on two symbols is $F_2 = S/\sim$ with concatenation as the product. The free group on n symbols can be defined similarly and is denoted F_n.

The free group on one symbol is isomorphic to $\mathbb{Z}$. However, it is the only free group that is abelian. It is common to use powers to denote a repeated symbol, thus $a^3 = aaa$, to denote inverses with negative exponents and use $1 = e$. If $n \neq m$ then F_n and F_m are not isomorphic.

If we take n copies of S^1 and join them together at a point the resulting space is call the *wedge of n circles*. The fundamental group is known to be F_n.



The fundamental group of the space above is F_5.

Fact. F_n us the "freest" group with n generators in the following sense. Let G be a group with n generators. Then there exists a homomorphism of F_n onto G.

Generators and Relations

Consider the equations, $a^2b^{-3} = 1$ and $c^2 = 1$. Suppose we want the "freest" group with three generators in which these equations, called *relations*, are true. Call the group G. Then, in G all "consequences" of these two relations must also be true. For example, $(a^2b^{-3})^2 = 1$ and $a^2c^8b^{-3} = 1$. Of course if we set $a = b = c = 1$, and thus make G to be the trivial group, all these equations are true, but this is not what we want. We do not want G to have any "extra" true equations.

To achieve this we proceed as follows. Let N be the smallest normal subgroup of F_3 that contains $\{a^2b^{-3}, c^2\}$. This is often denoted as $N = \langle\langle\{a^2b^{-3}, c^2\}\rangle\rangle$. Then we define $G = F_3/N$. Another notation for this is

$$G = \left\langle a, b, c \,|\, a^2b^{-3}, c^2 \right\rangle .$$

The generators are listed to the left of $|$, and the relations are listed on the right. Sometimes it is handy to write the relations as equations like this,

$$G = \left\langle a, b, c \,|\, a^2 = b^3, c^2 = 1 \right\rangle .$$

Think about why this makes sense. Every member of the normal subgroup N can be regarded as a consequence of the given relations and N is the largest subgroup with this property.

Examples. $\langle a \,|\, a^7 \rangle \cong \mathbb{Z}/7\mathbb{Z}$. $\langle a, b \,|\, ab = ba \rangle \cong \mathbb{Z}^2$.

Fact. Let $G = \langle a_1, \ldots, a_n \,|\, r_1, \ldots, r_m \rangle$. Let H be another group generated by $a_1, \ldots, a_n$ in which the equations $r_1 = 1$,...,$r_m = 1$ are true. Then there exists a homomorphism from G onto H.

Example. Let $T = S^1 \times S^1$. If we remove a point from T there is a deformation retraction of the resulting space to a figure-8, two circles joined at a point: $T - \{p\} \simeq S^1 \wedge S^1$. Thus, $\pi_1(T - \{p\}) \cong \pi_1(S^1 \wedge S^1)$. We have seen that $\pi_1(T) = \langle a, b \,|\, ab = ba \rangle$ and that $\pi_1(S^1 \wedge S^1) = \langle a, b \rangle$. If you look at the rectangle model of the torus you can see why removing a point removes the relation $ab = ba$.

Tietze Transformations

Definition. Let $S = \{a_1, \ldots, a_n\}$, $R = \{r_1, \ldots r_m\}$, and $G = \langle S \,|\, R \rangle$. The *Tietze transformations* are the following.

T1. Add a relation to R that is in $\langle\langle R \rangle\rangle$.

T2. Delete a relation from R that does not effect $\langle\langle R \rangle\rangle$.

T3. Simultaneously, add a new generator a_{n+1} to S and a new relation of the form $a_{n+1} = w \in F_n$ to R.

T4. Delete a generator a_i from S which only appears in one member of R which be be expressed as $a_i = w$, where w does not contain a_i, and delete this relation from R.

Theorem. [Heinrich Tietze, 1908] $\langle S_1 \,|\, R_1 \rangle \cong \langle S_2 \,|\, R_2 \rangle$ iff there is a finite sequence of Tietze transformations taking $\langle S_1 \,|\, R_1 \rangle$ to $\langle S_2 \,|\, R_2 \rangle$.

Example 1. Let $G = \langle a, b, c, d \,|\, ab = c, bc = d, cd = a, da = b \rangle$. Show that $G \cong \mathbb{Z}/5\mathbb{Z}$.

$$\begin{array}{rcll}
G &=& \langle a, b, c, d \,|\, ab = c, bc = d, cd = a, da = b \rangle & \\
&=& \langle a, b, c, d \,|\, ab = c, bc = d, cd = a, da = b, cbc = a \rangle & T1 \\
&=& \langle a, b, c, d \,|\, ab = c, bc = d, da = b, cbc = a \rangle & T2 \\
&=& \langle a, b, c, d \,|\, ab = c, bc = d, da = b, cbc = a, ca = 1 \rangle & T1 \\
&=& \langle a, b, c, d \,|\, ab = c, bc = d, cbc = a, ca = 1 \rangle & T2 \\
&\cong& \langle a, b, c \,|\, ab = c, cbc = a, ca = 1 \rangle & T4 \\
&=& \langle a, b, c \,|\, ab = c, cbc = a, ca = 1, bbab = 1 \rangle & T1 \\
&=& \langle a, b, c \,|\, ab = c, ca = 1, bbab = 1 \rangle & T2 \\
&=& \langle a, b, c \,|\, ab = c, bbab = 1, aba = 1 \rangle & T1, T2 \\
&\cong& \langle a, b \,|\, bbab = 1, aba = 1 \rangle & T4 \\
&=& \langle a, b \,|\, a = b^{-3}, aba = 1 \rangle & T1, T2 \\
&=& \langle a, b \,|\, a = b^{-3}, b^{-5} = 1 \rangle & T1, T2 \\
&\cong& \langle b \,|\, b^{-5} = 1 \rangle & T4 \\
&\cong& \langle b \,|\, b^{5} = 1 \rangle & T1, T2 \\
&\cong& \mathbb{Z}/5\mathbb{Z} &
\end{array}$$

Example 2. Let $G = \langle x, y \,|\, xyx = yxy \rangle$. Show that $G \cong \langle a, b \,|\, a^3 = b^2 \rangle$.

$$\begin{array}{rcll}
G &=& \langle x, y \,|\, xyx = yxy \rangle & \\
&\cong& \langle a, b, x, y \,|\, xyx = yxy, a = xy, b = yxy \rangle & T3, T3 \\
&=& \langle a, b, x, y \,|\, xyx = yxy, a = xy, b = yxy, a^3 = b^2 \rangle & T1 \\
&=& \langle a, b, x, y \,|\, a = xy, b = yxy, a^3 = b^2 \rangle & T2 \\
&=& \langle a, b, x, y \,|\, a = xy, y = ba^{-1}, a^3 = b^2 \rangle & T1, T2 \\
&=& \langle a, b, x, y \,|\, x = ay^{-1}, y = ba^{-1}, a^3 = b^2 \rangle & T1, T2 \\
&\cong& \langle a, b, y \,|\, y = ba^{-1}, a^3 = b^2 \rangle & T4 \\
&\cong& \langle a, b \,|\, a^3 = b^2 \rangle & T4
\end{array}$$

Definition. Let G_i be groups presented by $\langle S_i \,|\, R_i \rangle$ respectively for $i = 1, 2$. Assume $S_1 \cap S_2 = \phi$. Then the *free product* is defined by

$$G_1 * G_2 = \langle S_1 \cup S_2 \,|\, R_1 \cup R_2 \rangle .$$

Examples. $\langle a, b\rangle = \langle a\rangle * \langle b\rangle$. $\langle a, b \,|\, ab = ba\rangle * \langle c\rangle = \langle a, b, c \,|\, ab = ba\rangle \cong \mathbb{Z}^2 * \mathbb{Z}$.

The word problem. Given two words from the presentation of a group, how can we tell if they represent the same group member? For a free group this is easy - put each word into reduced form. But, there does not exist a general algorithm. [Novikov, 1955]

The isomorphism problem. Given two presentations, how can we decide if they represent the same group? That is, how can we tell if there is a sequence of Tietze moves taking one to the other? Unfortunately, we now know that there does not exist a general algorithm. [Adian, 1957; Rabin, 1958] However, for finitely generated abelian groups, there is an algorithm. We will cover it later.

For a history of these two topics see, *The word problem and the isomorphism problem for groups*, by John Stillwell, Bulletin (New Series) of the American Mathematical Society, Volume 6, Number 1, January 1982.