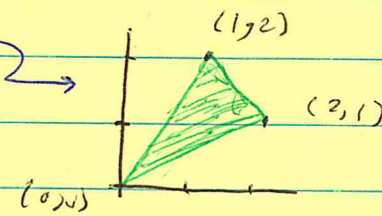
In the beginning there were Simplicies

Def Let $\{a_0, a_1, \dots, a_n\}$ be in $\mathbb{R}^N$ with $n \leq N$.
They are geometrically independent if the vectors $\overrightarrow{a_0a_1}, \overrightarrow{a_0a_2}, \dots, \overrightarrow{a_0a_n}$ are linearly independent in $\mathbb{R}^N$.

Def In this case the simplex σ spanned by $\{a_0, a_1, a_2, \dots a_n\}$ is the set

$$\sigma = \left\{ \sum_{i=0}^{n} t_i a_i \;\middle|\; t_i \geq 0, \ \sum t_i = 1 \right\}.$$

Ex $\{0, 2\}$ in $\mathbb{R}^1$ ~~gives~~ spans $\sigma = [0, 2]$.
$\{(0,0), (1,2), (2,1)\}$ in $\mathbb{R}^2$ spans $\longrightarrow$



Def Any simplex spanned by a nonempty subset of $\{a_0, a_1, \dots a_n\}$ is called a face of σ.

The face opposite a_j is $\left\{ \sum_{i \neq j} t_i a_i \;\middle|\; t_i \geq 0 \ \sum_{i \neq j} t_i = 1 \right\}$.

A face consisting of one point is called a vertex.

Fact Simplicies are compact convex and homeo. to the n-ball.

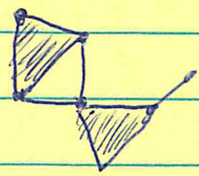
Notation The simplex spanned by $\{a_0, \dots a_n\}$ is denoted $[a_0, \dots, a_n]$.
For now the order does not matter.

Def A <u>simplicial complex</u> K in $\mathbb{R}^N$ is a set of simplicies in $\mathbb{R}^N$ s.t.

① Every face of a simplex of K is in K,

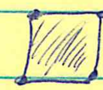
② The intersection of any two simplicies of K is a face of each or $\emptyset$.

Ex  K = seven simplices : 



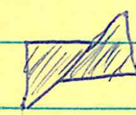
a simplicial complex

not a s.c.

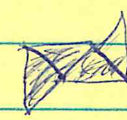


not a s.c.

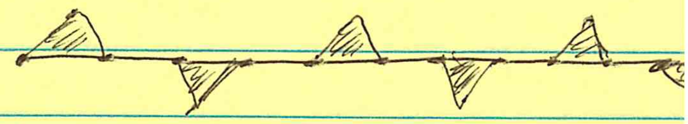
a s.c.

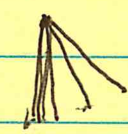


not a s.c.

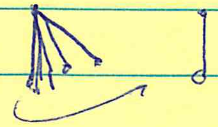


a s.c.

Infinite s.c.'s are allowed: 



Def $|K|$ = set of all points in members of K. But we give $|K|$ a topology that can be finer than the subsp. topology. It is determined by $A \subset |K|$ is closed iff $A \cap \sigma$ (in the subsp top) $\forall \sigma \in K$.

Ex  is not open in subsp top but is open in $|K|$ in the above top.

Fact If K is finite, then both topologies are the same.

Facts $|K|$ is Hausdorff.

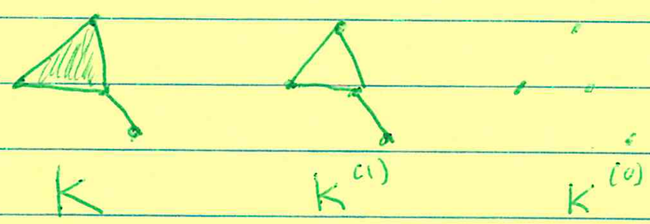
K finite $\Leftrightarrow$ $|K|$ is compact

K is "locally finite" $\Leftrightarrow$ $|K|$ is loc. compact.

K is loc. finite $\Leftrightarrow$ $|K|$ is metrizable (#7 in §2).

Def If $L \subset K$ and is a complex, we say it is a subcomplex of K

$K^{(p)}$ = all simplicies of K of dim p or less. called the p-skeleton of K.



Simplicial Maps

Def/Lemma (2.7) Let K and L be simplicial complexes. Let $f: K^{(0)} \to L^{(0)}$. Suppose that whenever a set of vertices $\{a_0, \dots a_j\}$ of K span a simplex of K the images $\{f(a_0), \dots, f(a_j)\}$ span a simplex of L. Then we can extend f to a continuous map

$$F: |K| \to |L| \quad \text{s.t.}$$

$$x = \sum_{i=0}^{j} t_i a_i \Rightarrow F(x) = \sum_{i=0}^{j} t_i f(a_i).$$

We call F the linear simplicial map induced by f.

Facts Compositions of l.s.m.'s are lsm's.

Def A lsm that is one-to-one and onto is a linear simplicial homeomorphism.

Note Simplicial complexes with linear simplicial maps form a catagory.

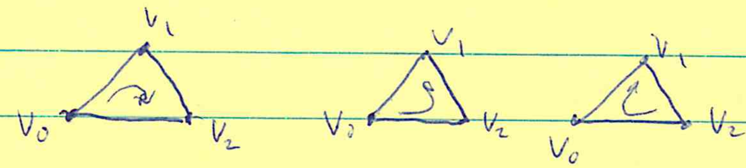
Homology Groups of a Simplicial Complex

We defined orientation of a simplex in order to make a group.

Intuitive Ex $\underset{v_0}{\bullet}\!\longrightarrow\!\underset{v_1}{\bullet} \equiv [v_0, v_1]$, $\underset{v_0}{\bullet}\!\longleftarrow\!\underset{v_1}{\bullet} \equiv [v_1, v_0]$, and $[v_1 v_0] = -[v_0, v_1]$.

Def Let σ be the simplex spanned by $\{v_0, \dots, v_n\}$. Then let $[v_0, v_1 \dots v_n]$ denote σ with an "orientation". If we permute two vertices we get the opposite orientation: $[v_0 v_1 v_2, \dots v_n] = -[v_1 v_0 v_2 v_3 \dots v_n]$, etc. There are always two orientation classes. For 0-simplices we will let $\pm v$ denote the two orientations.

Ex



$\sigma = [v_0 v_1 v_2]$ $\quad [v_0 v_2 v_1] = -\sigma$ $\quad [v_2 v_0 v_1] = --\sigma = \sigma$.

This extends our intuitive notation of clockwise and counter-clockwise to higher dimensions. Also, compare this with the det function: switching rows (or columns) changes the sign.

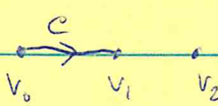
Def Let K be a simplicial complex. For $p = 0, 1, 2, 3 \ldots$ a p-chain is a function c from the set of oriented p-simplices of K to $\mathbb{Z}$ s.t.

(1) $c(-\sigma) = -c(\sigma)$ and (2) $c(\sigma) = 0$ for all but finitely many σ's of K.

The p-chains of K form a group under formal addition. The group is called $C_p(K)$. If $p < 0$ or $>$ the top dim of K, we let $C_p(K) =$ trivial gp.

$C_p(K)$ is always free abelian.

For any $c \in C_p(K)$ we may write $c = \sum n_i c_i$ where $n_i \in \mathbb{Z}$ and $c_i(\sigma_j) = \delta_{ij}$. That is $C_p(K)$ is generated by these c_i's.

Ex Let $K =$ 

$C_0(K)$ is generated by the 3 functions c_0^0, c_1^0, c_2^0.
In practice we abuse notation and write

$$C_0(K) = \langle v_0, v_1, v_2 \rangle = \{ n_0 v_0 + n_1 v_1 + n_2 v_2 \mid n_0, n_1, n_2 \text{ in } \mathbb{Z} \}.$$

So $C_0(K) \cong \mathbb{Z}^3$.

$C_1(K) = \langle e \rangle \cong \mathbb{Z}$.

Def We define the all important <u>boundary maps</u>.

Let $\partial_p : C_p(K) \to C_{p-1}(K)$ be

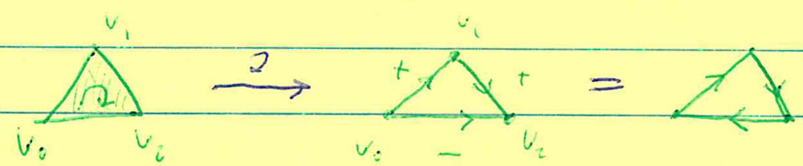
$$\partial_p [v_0 v_1 \dots v_p] = \sum_{i=0}^{p} (-1)^i [v_0 \dots \hat{v}_i \dots v_p]$$

↑ means deleted

for each p-simplex then extend to a group homomorphism. (If $C_p(K)$ is the trivial group we let ∂_p take 0 to 0.

Ex $\partial_1([v_0 v_1]) = v_1 - v_0$

$\partial_2([v_0 v_1 v_2]) = [v_1 v_2] - [v_0 v_2] + [v_0 v_1]$



$$\begin{aligned}\partial_1 \partial_2 ([v_0 v_1 v_2]) &= \partial_1([v_1 v_2]) - \partial_1([v_0 v_2]) + \partial_1([v_0 v_1]) \\ &= (v_2 - v_1) - (v_2 - v_0) + (v_1 - v_0) = 0.\end{aligned}$$

Lemma 5.3 $\partial_{p-1} \partial_p = 0$. Work through the proof of this.

Def Let $Z_p(K) = \ker \partial_p \subset C_p(K)$, called p-cycles

Let $B_p(K) = \mathrm{im}\, \partial_{p+1} \subset C_p(K)$, called p-boundaries.

By Lemma 5.3 $B_p(K) \subset Z_p(K)$ and it is easy to check it is a subgroup.

Let $H_p(K) = Z_p(K) / B_p(K)$, called p-th homology gp of K.

We will...

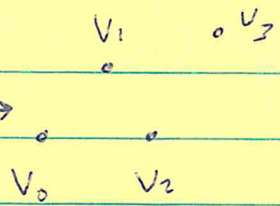
compute many examples,

show that linear simp. maps induce homomorphisms on the homology groups (think functor)

studies various homomorphism on sequences of homology groups, and

extend our definitions to define homology groups of more general top. spaces.

Ex 00 $K \to$



Find $H_0(K)$ and $H_1(K)$.

$C_0 = \langle v_0, v_1, v_2, v_3 \rangle \cong \mathbb{Z}^4$ $\quad C_i = 0$ for $i \neq 0$.

$Z_0 = \ker \partial_0 : C_0 \to 0 = C_0$

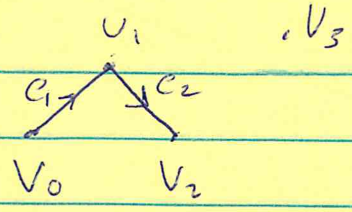
$B_0 = \operatorname{im} \partial_1 : 0 \to C_0 = 0.$

$H_0 = Z_0 / B_0 = Z_0 \cong \mathbb{Z}^4.$

$Z_1 = \ker \partial_1 : C_1 \to C_0 = 0$ since $C_1 = 0$.

$B_1 = \operatorname{im} \partial_2 : C_2 \to C_1 = 0$ since $C_1 = 0$.

$H_1 = Z_1 / B_1 = 0.$

Ex 0 K  Find $H_0(K)$ and $H_1(K)$.

$C_0 = \langle v_0, v_1, v_2, v_3 \rangle \cong \mathbb{Z}^4$

$C_1 = \langle e_1, e_2 \rangle \cong \mathbb{Z}^2$ $\qquad C_i = 0$ for $i \neq 0, 1$.

$Z_0 = \ker \partial_0 : C_0 \to 0 = C_0$

$B_0 = \operatorname{im} \partial_1 : C_1 \to C_0 = \langle \partial e_1, \partial e_2 \rangle = \langle v_1 - v_0, v_2 - v_1 \rangle$

To find Z_0/B_0 we use an alternative basis for Z_0.

$$Z_0 = \langle v_0, v_1, v_2, v_3 \rangle = \langle v_0, v_1 - v_0, v_2 - v_1, v_3 \rangle.$$

Then $Z_0/B_0 = \langle v_0 + B_0, v_3 + B_0 \rangle \cong \mathbb{Z}^2$.

Thus $H_0 \cong \mathbb{Z}^2$

$Z_1 = \ker \partial_1 : C_1 \to C_0 = ?$ Need $\partial(n_1 e_1 + n_2 e_2) = 0$

Thus, $n_1(v_1 - v_0) + n_2(v_2 - v_1) = 0$.

or, $-n_1 v_0 + (n_1 - n_2) v_1 + n_2 v_2 = 0$.

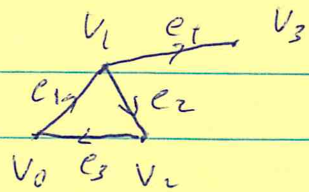
Thus $n_1 = n_2 = 0$ is the only solution

Hence $Z_1 = 0$.

$B_1 = \operatorname{im} \partial_2 : 0 \to C_1 = 0$.

Thus, $H_1 = Z_1 / B_1 = 0$.

Ex 1 K



Find $H_0(K)$ and $H_1(K)$

$C_0 = \langle v_0, v_1, v_2, v_3 \rangle \cong \mathbb{Z}^4$

$C_1 = \langle e_1, e_2, e_3, e_4 \rangle \cong \mathbb{Z}^4 \qquad C_i = 0 \quad i \neq 0, 1.$

$Z_0 = \ker \partial_0 : C_0 \to 0 = C_0.$

$B_0 = \operatorname{im} \partial_1 : C_1 \to C_0 = \langle \partial e_1, \partial e_2, \partial e_3, \partial e_4 \rangle$

$\qquad = \langle v_1 - v_0,\ v_2 - v_1,\ v_0 - v_2,\ v_3 - v_1 \rangle.$

You might think $B_0 \cong \mathbb{Z}^4$ but this is not true.

Notice

$$(v_2 - v_1) + (v_0 - v_2) = -(v_1 - v_0).$$

So we can drop the first generator.

$$B_0 = \langle v_2 - v_1,\ v_0 - v_2,\ v_3 - v_1 \rangle.$$

To find Z_0 / B_0 we use an alternative basis for Z_0

$$Z_0 = \langle v_0,\ v_2 - v_1,\ v_0 - v_2,\ v_3 - v_0 \rangle$$

Now $Z_0 / B_0 = \langle v_0 + B_0 \rangle \cong \mathbb{Z}$

Thus $H_0(K) \cong \mathbb{Z}$.

$Z_1 = \ker \partial_1 : C_1 \to C_0$

Let $n_1 e_1 + n_2 e_2 + n_3 e_3 + n_4 e_4 \in C_1$

When does $\partial_1 (n_1 e_1 + n_2 e_2 + n_3 e_3 + n_4 e_4) = 0$?

$$n_1 (v_1 - v_0) + n_2 (v_2 - v_1) + n_3 (v_0 - v_2) + n_4 (v_3 - v_1) = 0$$

$$(n_3 - n_1) v_0 + (n_1 - n_2 - n_4) v_1 + (n_2 - n_3) v_2 + n_4 v_3 = 0.$$

Then $n_4 = 0$ and $n_1 = n_2 = n_3$. Let $n = n_1 = n_2 = n_3$.

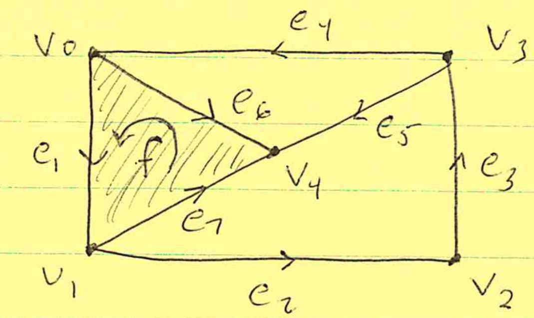
Then

$$\ker \partial_1 = \{ n(e_1 + e_2 + e_3) \mid n \in \mathbb{Z} \} \cong \mathbb{Z}.$$

~~Thus H_1~~

$B_1 = \operatorname{im} \partial_2 : 0 \to C_1 = 0$. Thus $H_1 = Z_1 / B_1 = Z_1 \cong \mathbb{Z}$.

Ex 2



This is K.

Find $H_0(K)$, $H_1(K)$, $H_2(K)$.

$C_0 = \langle v_0, v_1, v_2, v_3, v_4 \rangle \cong \mathbb{Z}^5$.
$C_1 = \langle e_1, e_2, e_3, e_4, e_5, e_6, e_7 \rangle \cong \mathbb{Z}^7$.
$C_2 = \langle f \rangle \cong \mathbb{Z}$.

H_0

$Z_0 = \ker \partial_0 : C_0 \to 0 = C_0$.
$B_0 = \operatorname{im} \partial_1 : C_1 \to C_0$. B_0 is generated by

$\partial_1 e_1 = v_1 - v_0$ $\quad$ $\partial_1 e_5 = v_4 - v_3$
$\partial_1 e_2 = v_2 - v_1$ $\quad$ $\partial_1 e_6 = v_4 - v_0$
$\partial_1 e_3 = v_3 - v_2$ $\quad$ $\partial e_7 = v_4 - v_1$
$\partial_1 e_4 = v_0 - v_3$

We immediately notice the following

$\partial e_4 = \partial_1 e_1 + \partial_1 e_2 + \partial_1 e_3$. So we drop it.
$\partial e_5 = \partial e_6 + \partial e_7$. So we drop it.
$\partial e_6 = \partial e_7 + \partial e_1$. So we drop it.

Thus, $B_0 = \langle v_1 - v_0, v_2 - v_1, v_3 - v_2, v_4 - v_1 \rangle$.

Each involves a unique vertex, so they are independent and we have a basis.

Now $Z_0 = C_0$ can be rewritten as

$$\langle v_0, v_1 - v_0, v_2 - v_1, v_3 - v_2, v_4 - v_1 \rangle.$$

Thus, $\quad H_0 = Z_0/B_0 = \langle v_0 + B_0 \rangle \cong \mathbb{Z}.$

<u>H_1</u> $\quad Z_1 = \ker \partial_1 : C_1 \to C_0$. When is $\partial_1 \left(\sum_{i=1}^{7} n_i e_i \right) = 0$?
We need to solve

$$n_1(v_1 - v_0) + n_2(v_2 - v_1) + n_3(v_3 - v_2) + n_4(v_0 - v_3) + n_5(v_4 - v_3) + n_6(v_4 - v_0) + n_7(v_4 - v_1) = 0.$$

This is

$$(-n_1 + n_4 - n_6)v_0 + (n_1 - n_2 - n_7)v_1 + (n_2 - n_3)v_2 + (n_3 - n_4 - n_5)v_3 + (n_5 + n_6 + n_7)v_4 = 0.$$

In matrix form we have

$$\begin{bmatrix} -1 & 0 & 0 & 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \\ n_6 \\ n_7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} -1 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 & -1 & -1 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & -1 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{matrix} \\ \\ +R_2 \\ \\ \\ \end{matrix}$$

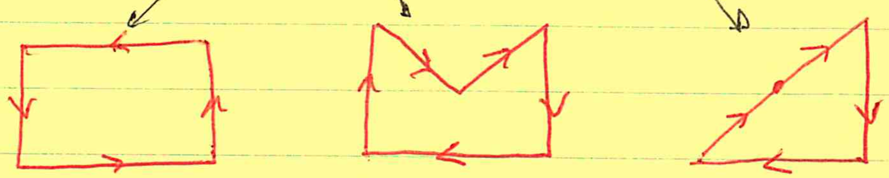
$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{matrix} \\ (-1) \\ \\ +R_3 \\ \\ \end{matrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \\ \\ (-1) \\ +R_4 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \\ +R_4 \\ \\ \\ \end{matrix}$$

$$\begin{aligned} n_1 &= n_4 - n_6 \\ n_2 &= n_4 - n_6 - n_7 \\ n_3 &= n_4 - n_6 - n_7 \\ n_4 &= n_4 \\ n_5 &= -n_6 - n_7 \\ n_6 &= n_6 \\ n_7 &= n_7 \end{aligned}$$

– free

Thus,
$$\begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \\ n_6 \\ n_7 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{z_1} n_4 + \underbrace{\begin{bmatrix} -1 \\ -1 \\ -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}}_{z_2} n_6 + \underbrace{\begin{bmatrix} 0 \\ -1 \\ -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}}_{z_3} n_7$$



Thus $Z_1 \cong \mathbb{Z}^3$ and $Z_1 = \langle z_1, z_2, z_3 \rangle$ where

$z_1 = e_1 + e_2 + e_3 + e_4$

$z_2 = -e_1 - e_2 - e_3 - e_5 + e_6$

$z_3 = -e_2 - e_3 - e_5 + e_7$

Now $B_1 = \operatorname{im} \partial_2 : C_2 \to C_1$, $B_1 = \langle \partial_2 f \rangle = \langle e_1 + e_7 \cancel{+} e_6 \rangle$

Let $z_4 = e_1 \cancel{+} e_6 + e_7$.

To compute Z_1 / B_1 we change the basis for Z_1 as follows.

Notice $z_4 = z_3 - z_2$.

Thus, $Z_1 = \langle z_1, z_4, z_3 \rangle$.

Finally, $H_1 = Z_1 / B_1 = \dfrac{\langle z_1, z_4, z_3 \rangle}{\langle z_4 \rangle} = \langle z_1 + B_1, z_3 + B_1 \rangle \cong \mathbb{Z}^2$.

H_2

$Z_2 = \ker \partial_2 : C_2 \to C_1$.

The members of C_2 are of the form nf. Solving $\partial(nf) = 0$ gives $n(e_1 + e_7 - e_6) = 0$ so $n = 0$. Thus $Z_2 = 0$.

$B_2 = \operatorname{im} \partial_3 : 0 \to C_2 = 0$.

Thus $H_2(K) = 0/0 = 0$.