An Outline of Chapter 2

The main goal ~~are~~ is to prove the topological invariance of homology groups: if K and L are complexes ~~with~~ s.t. $\exists$ a homeo $h : |K| \to |L|$ then $h_* : H_p(K) \to H_p(L)$, $\forall p$, are isomorphisms. This is Cor. 18.2.

Another important goal is to prove that if $h, k : |K| \to |L|$ are homotopic maps then $h_* = k_*$. This is Thm 19.2.

The main tool is simplicial approximations. If we ~~refine~~ subdivide the simplicies enough we can find a simp map that approximates $h : |K| \to |L|$.

§14: Simplicial Approximations

<u>Def</u> (From page 11) Let K be a simp. complex and v be a vertex of K. Then $st(v)$ = union of interiors of simplicies of that have v as a vertex.

<u>Def</u> Let $h: |K| \to |L|$ be cont. (K and L are s. complexes) If $f: K \to L$ is a simp. map s.t.

$$h(st(v)) \subset st(f(v))$$

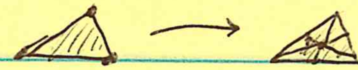
$\forall$ vertices v of K, then f is a <u>simplicial approx</u> of h.

<u>Lemma</u> Any two simplicial approximations of h are contiguous.

§15 Barycentric Subdivisions.

<u>Def</u> Let K be a simp. comp. A simp. comp. K' is a <u>subdivision</u> of K if

1. Each simplex of K' is contained is a simplex of K
2. Each simplex of K equals a union of finitely many simplicies of K'.

Ex Barycentric subdivision: 

$K \longrightarrow sd\,K.$

Thm 15.4 Let K be a finite simp complex. Put a metric on $|K|$, $\forall \varepsilon > 0$ $\exists N$ s.t. $\sigma \in sd^N K \Rightarrow diam(\sigma) < \varepsilon$.

Thm 16.5 Let $h: |K| \to |L|$ be cont. $\exists$ a subdivision K' of K s.t. $\exists$ a simplicial approx. $f: K' \to L$ of h.

Thm 17.2 Let K' be a subdivision of K. $\exists!$ aug-pres. chain map

$$\lambda: \mathcal{C}(K) \to \mathcal{C}(K')$$

that does just what is needed.

Def We call λ the subdivision operator.

Thm 18.1 Let $h: |K| \to |L|$ be cont.
Let K' be a subdivision of K.
Let $f: K' \to L$ be a simp. approx. to h.
Let λ be the subdivision operator.
We have the induced homomorphisms:

$$\lambda_*: H_p(K) \to H_p(K') \text{ and } f_*: H_p(K') \to H_p(L).$$

Let $h_* = f_* \circ \lambda_*: H_p(K) \to H_p(L)$.

Then $*(h: |K| \to |L|) \Longrightarrow h_*: H_p(K) \to H_p(L)$ is a functor.

Cor 18.2 If h is a homeomorphism $: |K| \to |L|$, then $h_* : H_p(K) \to H_p(L)$ is a isomorphism. This works for reduced homology too.

Thm 18.3 Let $h: |K| \to |L|$ be a homeo and $h: |K_0| \to |L_0|$ be onto. Then

$$H_p(K, K_0) \cong H_p(L, L_0).$$

§19 Homotopy

Recall $f, g : X \to Y$ are *homotopic*, written $f \simeq g$, if

$$\exists\, H : X \times I \xrightarrow{\text{cont}} Y \text{ s.t. } H(x,0) = f(x) \text{ and } H(x,1) = g(x).$$

X and Y are *homotopically equivalent* if $\exists\, f: X \to Y, g: Y \to X$ s.t. $f \circ g \simeq id_Y$, $g \circ f \simeq id_X$.

Thm 19.2 If $h, k : |K| \to |L|$ are homotopic, then $h_* = k_*$ on homology and reduced homology groups.

Thm 19.3 Analog holds for relative homology groups.

Thm 19.5 $|K| \simeq |L| \Rightarrow H_p(K) \cong H_p(L)$ and $\tilde{H}_p(K) \cong \tilde{H}_p(L)$.

Note: The main technical issue is how to find a simp. complex whose polytope (underlying space) is $|K| \times I$.