§10

Homology with Coefficients not $\mathbb{Z}$.

We skipped §10 before. I'll cover here, but use CW complexes instead of simplicial complexes for the calculations.

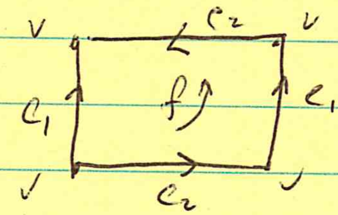
Def Let X be a CW complex. Let G be an abelian gp. Then

$$D_p(X;G) = \left\{ \sum_{\text{finite}} g_i c_i \;\middle|\; g_i \in G,\ c_i \text{ p-cells} \right\}.$$

We define ∂ as before. Then $Z_p(X;G) = \ker \partial_p$, $B_p(X;G) = \operatorname{im} \partial_{p+1}$ and $H_p(X;G) = Z_p / B_p$.

Ex Recall for the Klein bottle $H_0 \cong \mathbb{Z}$, $H_1 \cong \mathbb{Z} \oplus \mathbb{Z}/2$, and $H_2 \cong 0$. Find $H_*(K;\mathbb{Q})$.

Sol For K use



$D_0 = \langle v \rangle_{\mathbb{Q}} = \{ qv \mid q \in \mathbb{Q} \} \cong \mathbb{Q}.$

$D_1 = \langle e_1, e_2 \rangle_{\mathbb{Q}} = \{ q_1 e_1 + q_2 e_2 \mid q_1, q_2 \in \mathbb{Q} \} \cong \mathbb{Q}^2.$

$D_2 = \langle f \rangle_{\mathbb{Q}} = \{ qf \mid q \in \mathbb{Q} \} \cong \mathbb{Q}.$

$$0 \to D_2 \xrightarrow{\partial_2} D_1 \xrightarrow{\partial_1} D_0 \xrightarrow{\partial_0} 0$$

$\partial_1 e_1 = v - v = 0$ and $\partial_1 e_2 = v - v = 0$. So $\operatorname{im} \partial_1 = 0$.
Thus,

$$H_0(K;Q) = \frac{D_0}{0} = D_0 \cong Q.$$

$\ker \partial_1 = D_1$. $\operatorname{im} \partial_2 = \langle \partial f \rangle_Q = \langle e_1 + e_2 \mp e_1 + e_2 \rangle_Q$

$$= \langle 2e_2 \rangle_Q = \{ q2e_2 \mid q \in Q \}$$

$$\cong \{ qe_2 \mid q \in Q \}$$

$$H_1(K;Q) = \frac{\langle e_1, e_2 \rangle_Q}{\langle e_2 \rangle_Q} \cong \langle e_1 \rangle_Q \cong Q.$$

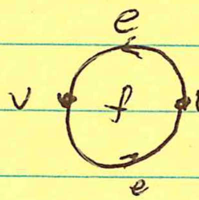
$H_2 = \frac{\ker \partial_2}{0}$ $\quad$ $\partial_2 qf = 2qe_1 = 0$ iff $q = 0$.

Thus $\ker \partial_2 \cong 0$ ~~$\langle e_2 \rangle_Q \cong Q$~~.

Thus $H_2(K;Q) \cong 0$.

Ex Find $H_i(P^2; \mathbb{Z}/n)$ for $i = 0, 1, 2$, $n = 3, 4$.

$n = 3$



$= P^2$ as a CW complex.

$D_0 = \langle v \rangle \cong \mathbb{Z}/3$.

$D_1 = \langle e \rangle \cong \mathbb{Z}/3$

$D_2 = \langle f \rangle \cong \mathbb{Z}/3$.

$\partial v = 0$, $\partial e = 0$ and $\partial f = 2e$.

$$0 \to D_2 \to D_1 \to D_0 \to 0$$

$$H_0 = \frac{D_0}{0} = D_0 \cong \mathbb{Z}/3.$$

$$H_1 = \frac{\ker \partial_1}{\operatorname{im} \partial_2} = \frac{\langle e \rangle}{\langle 2e \rangle} \cong \frac{\mathbb{Z}/3\mathbb{Z}}{2 \times (\mathbb{Z}/3\mathbb{Z})} = \frac{\mathbb{Z}/3}{\mathbb{Z}/3} = 0.$$

?

$(0, 1, 2) \xrightarrow{\times 2} (0, 2, 1)$

$$H_2 = \frac{\ker \partial_2}{0}$$

$\ker \partial_2 = 0$ since $2n = 0 \bmod 3$ only when $n = 0$.

Thus, $H_2 = 0$.

$\partial nf = 2ne = 0 \Rightarrow 2n = 0 \bmod 3$

$D_2 \cong \mathbb{Z}/4$, $D_1 \cong \mathbb{Z}/4$, $D_0 \cong \mathbb{Z}/4$.

$A = 4$

$H_0 = \frac{D_0}{0} \cong \mathbb{Z}/4$

$H_1 = \frac{\ker \partial_1}{\operatorname{im} \partial_2}$ $\quad \ker \partial_1 = D_1$ since $\partial e = 0$.

$\operatorname{im} \partial_2 = 2 \times (\mathbb{Z}/4)$

$(0,1,2,3) \xrightarrow{\times 2} (0,2,0,2)$

Thus $H_1 \cong \frac{(0,1,2,3)}{(0,2)} \cong \mathbb{Z}/2$

$H_2 = \frac{\ker \partial_2}{0}$. $\quad \partial_2 f = 2e$.

$\partial_2 nf = 0 \quad (n \in \mathbb{Z}/4)$

$\Rightarrow 2ne = 0e$

$2n = 0$.

So, $n = 0$ or $n = 2$.

Thus,

$H_2 \cong \mathbb{Z}/2$.

Summary

$H_0(P^2; \mathbb{Z}/3) \cong \mathbb{Z}/3$ $\quad H_0(P^2; \mathbb{Z}/4) \cong \mathbb{Z}/4$

$H_1(P^2; \mathbb{Z}/3) = 0$ $\quad H_1(P^2; \mathbb{Z}/4) \cong \mathbb{Z}/2$

$H_2(P^2; \mathbb{Z}/3) = 0$ $\quad H_2(P^2; \mathbb{Z}/4) \cong \mathbb{Z}/2$

See the exercises at the end of §10 for more examples.