Lecture Notes for Section 12

Part I. Simplicial maps and induced homomorphisms.
Part II. Contiguous simplicial maps and chain homotopies.
Part III. Extend this to relative homology groups [read on your own].

Part I

DEFINITION. (From Section 2) Let K and L be complexes. Let f be a function from the vertices of K to the vertices of L. While f need not be one-to-one or onto we assume that the vertices of each simplex in K are taken onto the vertices of some simplex of L so that we can extend f linearly to a continuous function $f : |K| \to |L|$. We abuse notation and write $f : K \to L$ for the *simplicial map* from K to L determined by the original vertex map.

DEFINITION. Let $f : K \to L$ be a simplicial map. We define the *induced homomorphism* on the chain groups

$$f_{\#} : C_p(K) \to C_p(L)$$

as follows. Let $\sigma = [v_0, \ldots, v_p]$ be a p-simplex of K. Then $\{f(v_0), \ldots, f(v_p)\}$ spans a p'-simplex of L where $p' \leq p$. Let

$$f_{\#}(\sigma) = \begin{cases} [f(v_0), \ldots, f(v_p)] & \text{if} \quad p' = p \\ 0 & \text{if} \quad p' < p. \end{cases}$$

We can extend $f_{\#}$ to a homomorphism since $f_{\#}(-\sigma) = -f_{\#}(\sigma)$ is clear.

LEMMA (12.1). $f_{\#}$ commutes with ∂.

Proof. See textbook.

COROLLARY. $f_{\#}$ induces a homomorphism on homology groups,

$$f_* : H_p(K) \to H_p(L).$$

Proof. Let $c \in H_p(K)$. Let z and z' be representative p-cycles of c, that is $c = z + B_p(K) = z' + B_p(K)$ for $z, z' \in Z(p(K)$. We will show that $f_{\#}(z) \sim f_{\#}(z')$ This means we can define $f_*(c) = f\#(z)$ for any $z \in c$ without ambiguity.

(1) We claim $f_{\#}(z) \in Z_p(L)$. *Proof.* $\partial f_{\#}(z) = f_{\#}\partial(z) = f_{\#}(0) = 0$. Likewise $f_{\#}(z') \in Z_p(L)$.

(2) We claim $f_{\#}(z) \sim f_{\#}(z')$. Let $b = z - z'$. Since $z \sim z'$ we know $b \in B_p(K)$. Hence $\exists\, d \in C_{p+1}(K)$ with $\partial d = b$. Now

$$f_{\#}(b) = f_{\#}(\partial d) = \partial f_{\#}(d) \in B_p(L).$$

Thus, $f_{\#}(z) - f_{\#}(z') = f_{\#}(b) \in B_p(L)$ proving the claim.

Thus, we can define $f_*(c) = f_\#(z)+B_p(L)$ for any choice of $z \in C$ without ambiguity. $\square$

THEOREM (12.2). Let $*(f : K \to L) = \{f_* : H_p(K) \to H_p(L) \,|\, \forall\, p\}$. Then $*$ is a functor.

Outline of Proof. (a) Let id $: K \to K$ be the identity simplicial map. Then one checks that $\mathrm{id}_* : H_p(K) \to H_p(K)$ is the identity isomorphism $\forall\ p$.

(b) Let $f : K \to L$ and $g : L \to M$ be simplicial maps. Then one checks that $(g \circ f)_* = g_* \circ f_*$. $\square$

FACT. This all works for reduced homology groups. See textbook.

Part II.

Definition. Let $f, g : K \to L$ be simplicial maps. We say f and g are *contiguous* if $\forall \sigma = [v_0, \ldots, v_p] \in K$ the set

$$\{f(v_0, \ldots, f(v_p), g(v_0), \ldots, g(v_p)\}$$

spans a simplex in L.

Remark. Contiguous is a standard English word that mean next to.

Example. In the figure below define $f : K \to L$ by

$$f(v_0) = w_0 \quad f(v_1) = w_1 \quad f(v_2) = w_2$$

and $g : K \to L$ by

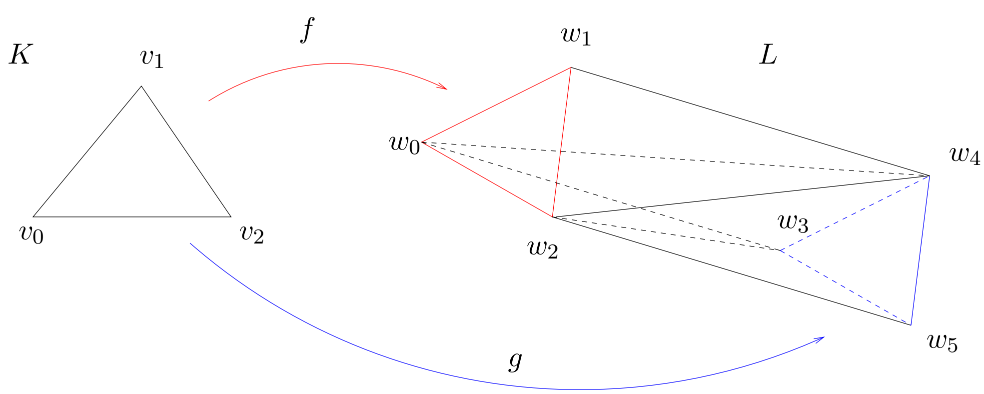
$$g(v_0) = w_3 \quad g(v_1) = w_4 \quad g(v_2) = w_5.$$

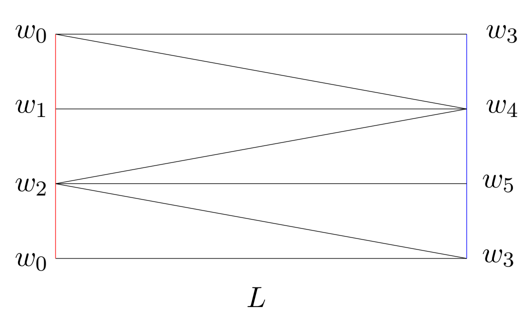
For each v_i the set $\{f(v_i), g(v_i)\}$ spans an edge of L. But for $\sigma = [v_0, v_1]$ we have the set

$$\{f(v_0), f(v_1), g(v_0), g(v_1)\} = \{w_0, w_1, w_3, w_4\}$$

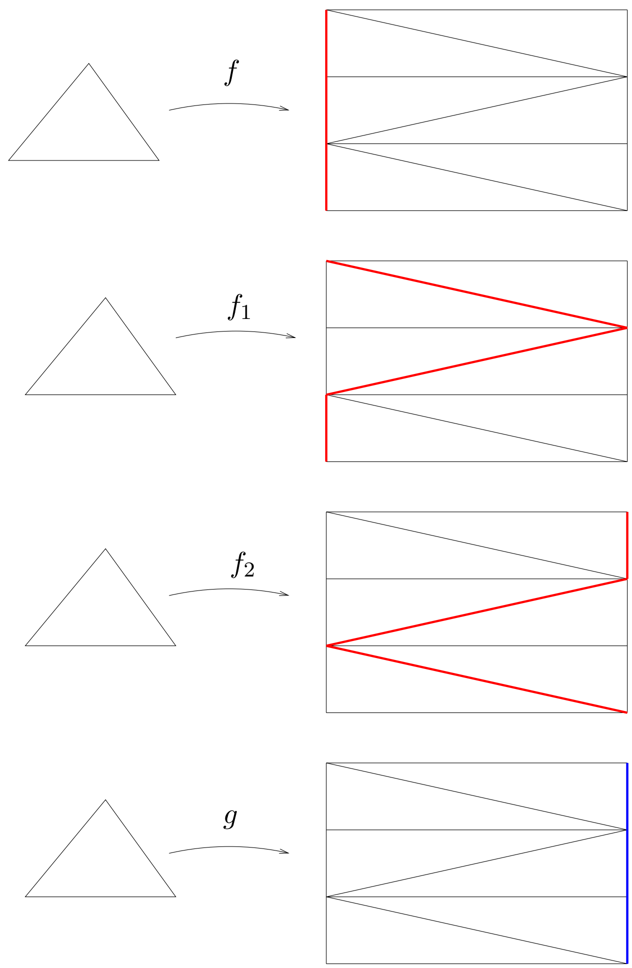
and these do not span a simplex of L.

Thus, f and g are not contiguous.





But, consider the following simplicial maps of K into L.



Now f is contiguous to f_1 which is contiguous to f_2 which is contiguous to g.

Definition. Whenever this happens we will say that f and g are *eventually contiguous.*

Remark. Think of this as a discrete/combinatorial analog of homotopy.

Notice that in this example $f_* = g_*$. We will show that whenever f and g are eventually contiguous this happens.

The next definition will seem unnatural at first.

Definition. Let $f, g : K \to L$ be simplicial maps. Suppose $\forall\, p\ \exists$ a homomorphism $D : C_p(K) \to C_{p+1}(L)$ s.t.

$$\partial D + D\partial = g_\# - f_\#.$$

Such a D is called a *chain homotopy* between f and g. When this happens we say f and g are *chain homotopic.*

Theorem (12.4). If f and g are chain homotopic then $f_* = g_*$.

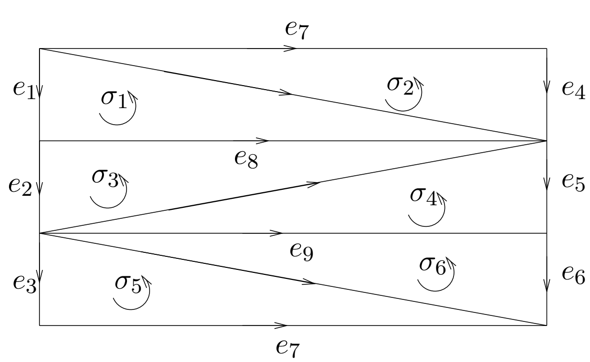
Proof. Let $z \in Z_p(K)$. Then

$$g_\#(z) - f_\#(z) = \partial Dz + D\partial z = \partial Dz \in B_p(L).$$

Thus, $g_\#(z) \sim f_\#(z)\ \forall\, z \in Z_p(K)$, so $g_* = f_*$. □

Return to Example.

We shall construct D. Really all we need is to have $D : Z_p(K) \to C_{p+1}(L)$ be s.t. $\partial D(z) = g_\#(x) - f_\#(z)$. But this is hard to do because we would need need to chose a basis for $Z_p(K)$ and define D on these basis cycles and then extend to $Z_p(K)$. Instead we define D on all of $C_p(K)$ by defining D on p-simplicies. This is easier, but the "cost" is the $D\partial$ term in the definition. But, $D\partial$ is always 0 on cycles and so causes on harm.



For $p = 0$, let $D(v_i) =$ the edge $[f_\#(v_i), g_\#(v_i)] \in C_1(L)$.

For $p = 1$, let $D([v_0, v_1]) = -(\sigma_1 + \sigma_2)$ in $C_2(L)$,
$D([v_1, v_2]) = -(\sigma_3 + \sigma_4)$, and
$D([v_2, v_0]) = -(\sigma_5 + \sigma_6)$.

Check for $p = 0$: Let $c \in C_0(K)$, $c = \sum n_i v_i$. Then

$$\partial Dc = \sum n_i \partial D v_i = \sum n_i (g_\#(v_i) - f_\#(v_i))$$
$$= g_\#(c) - f_\#(c) = g_\#(c) - f_\#(c) - D\partial c,$$

since $D\partial c = 0$.

Check for $p=1$: Let $c = n_1[v_0,v_1]+n_2[v_1,v_2]+n_3[v_2,v_0] \in C_1(K)$. Then

$$\partial Dc = -n_1\partial(\sigma_1+\sigma_2) - n_2\partial(\sigma_3+\sigma_4) - n_3\partial(\sigma_5+\sigma_6)$$

$$= -n_1(e_1+e_8-e_7-e_4) - n_2(e_2-e_8+e_9-e_5) - n_3(e_3-e_9+e_7-e_6)$$

$$= -(n_1e_1+n_2e_2+n_3e_3)+(n_1e_4+n_2e_5+n_3e_6)+(n_2-n_1)e_8+(n_3-n_2)e_9+(n_1-n_3)e_7.$$

Now,

$$-f_\# c = -(n_1e_1+n_2e_2+n_3e_3) \quad g_\# c = n_1e_4+n_2e_5+n_3e_6,$$

and

$$\begin{aligned} D\partial c &= D(n_1(v_1-v_0)+n_2(v_2-v_1)+n_3(v_0-v_2)) \\ &= D\big((n_3-n_1)v_0+(n_1-n_2)v_1+(n_2-n_3)v_2\big) \\ &= (n_3-n_1)e_7+(n_1-n_2)e_8+(n_2-n_3)e_9. \end{aligned}$$

Thus,

$$\partial Dc + D\partial c = g_\# c - f_\# c.$$

In general, the construction of D can be quite ad hoc.

Theorem (12.5). If $f, g : K \to L$ are eventually contiguous then $\exists$ a chain homotopy between f and g, and hence $f_* = g_*$.

Proof. It is enough to prove this for f and g contiguous.

Let $\sigma = [v_0,\dots,v_p] \in K$. Let $L(\sigma) =$ the subcomplex of L whose vertex set is

$$\{f(v_0),\dots,f(v_p),g(v_0),\dots,g(v_p)\};$$

and $L(\sigma)$ contains all faces formed from these.

We have the following facts.

(1) $L(\sigma)$ is not empty and it is acyclic.

(2) If s is a face of σ, then $L(s) \subset L(\sigma)$.

(3) $\forall\ \ \sigma \in K$, the chains $f_\#(\sigma)$ and $g_\#(\sigma)$ are carried by $L(\sigma)$.

We must show a chain-homotopy D exists.

Let $p=0$. $\exists$ a 1-chain c in $L(v)$ from $f_\#(v)$ to $g_\#(v)$ for each vertex v of K since f and g are contiguous. Define $D(v)=c$. Then

$$\partial Dv + D\partial v = \partial c + 0 = g_\#(v) - f_\#(v).$$

Note that $D(v)$ is carried by $L(v)$.

Now, assume D is defined for all dimensions $< p$ and is such that $\forall$ simplicies s of dimension $< p$ we have that $D(s)$ is carried by $L(s)$ and

$$\partial Ds + D\partial s = g_{\#}(s) - f_{\#}(s).$$

Let σ be a p-simplex of K. Let

$$c = g_{\#}(\sigma) - f_{\#}(\sigma) - D\partial c.$$

Claim 1: c is a cycle. (See textbook.)

Claim 2: c is carried by $L(\sigma)$. (See textbook.)

Since $L(\sigma)$ is acyclic we know c is a boundary of some $p+1$-chain in $L(\sigma)$. Let d be just such a $p+1$-chain. Define $D\sigma = d$. Then

$$\partial D\sigma = \partial d = c = g_{\#}(\sigma) - f_{\#}(\sigma) - D\partial\sigma.$$

Thus,

$$\partial D\sigma + D\partial\sigma = g_{\#}(\sigma) - f_{\#}(\sigma).$$

We can extend D to p-chains. $\square$

Part III

All this can be extended to relative homology groups. Read this on your own in the textbook.