Section 13 Acyclic Carriers

Def Let $\mathcal{C} = \{(C_p, \partial_p)\}_{p\in\mathbb{Z}}$ where each C_p is an abelian gp and $\partial_p : C_p \to C_{p-1}$ is a homomorphism. If $\partial\circ\partial = 0$ we say $\mathcal{C}$ is a chain complex. If $C_p = 0$ $\forall p<0$, then $\mathcal{C}$ is non-negative. If each C_p is free, then $\mathcal{C}$ is free.

Def Let $\mathcal{C}$ be a non-neg. ch. complex. Suppose $\exists\, \varepsilon : C_0 \to \mathbb{Z}$ a hom. that is onto and $\varepsilon\circ\partial_1 = 0$. Define

$$\mathcal{C}_\varepsilon = \{\ldots (C_p, \partial_p) \ldots (C_1, \partial_1), (C_0, \varepsilon), (\mathbb{Z}, 0), (0,0) \ldots\}.$$

Then $\mathcal{C}_\varepsilon$ is called an augmentation of $\mathcal{C}$.

Def $H_p(\mathcal{C}) = \dfrac{\ker \partial_p}{\operatorname{im} \partial_{p+1}}$ and $\tilde{H}_p(\mathcal{C}) = H_p(\mathcal{C}_\varepsilon)$.

Fact $H_p(\mathcal{C}) = \tilde{H}_p(\mathcal{C})$ for $p \neq 0$ and $H_0(\mathcal{C}) = \tilde{H}_0(\mathcal{C}) \oplus \mathbb{Z}$.

Def Let $\mathcal{C} = \{C_p, \partial_p\}$ and $\mathcal{C}' = \{C'_p, \partial'_p\}$ be chain complexes. Let $\phi = \{\phi_p : C_p \to C'_p\}_{p \in \mathbb{Z}}$ be s.t.

$$\partial'_p \phi_p = \phi_{p-1} \partial_p \qquad \forall p.$$

Then ϕ is called a chain map. We write $\phi : \mathcal{C} \to \mathcal{C}'$.

Picture

$$\begin{array}{ccccccc}
\to & C_p & \xrightarrow{\partial_p} & C_{p-1} & \xrightarrow{\partial_{p-1}} & C_{p-2} & \to \cdots \\
 & \downarrow \phi_p & & \downarrow \phi_{p-1} & & \downarrow \phi_{p-2} & \\
\to & C'_p & \xrightarrow[\partial'_p]{} & C'_{p-1} & \xrightarrow[\partial'_{p-1}]{} & C'_{p-2} & \to \cdots
\end{array}$$

Fact It is easy to check that a chain map $\phi : \mathcal{C} \to \mathcal{C}'$ induces a hom. $\phi_* = \{\phi_{*,p} : H_p(\mathcal{C}) \to H_p(\mathcal{C}')\}$ and that

① $id \to id_*$

② $(\psi \circ \phi)_* = \psi_* \circ \phi_*$

Def Let $\mathcal{C}, \mathcal{C}'$ be chain complexes, with augmentations $\mathcal{C}_\varepsilon, \mathcal{C}'_{\varepsilon'}$. Let $\phi : \mathcal{C} \to \mathcal{C}'$ be a ch. map and suppose $\varepsilon' \circ \phi_0 = \varepsilon$. (We say ϕ is augmentation preserving.) We can extend ϕ to $\tilde{\phi} : \mathcal{C}_\varepsilon \to \mathcal{C}'_{\varepsilon'}$ as follows. Define $\tilde{\phi}_p = \phi_p$ for $p \neq -1$ and let $\tilde{\phi}_{-1} : \mathbb{Z} \to \mathbb{Z}$ be the id.

Then $\tilde{\phi}$ induces a hom. on reduced homology

$$\tilde{\phi}_* : H_p(\mathcal{C}_\varepsilon) \to H_p(\mathcal{C}'_{\varepsilon'}).$$

Def Let $\phi, \psi : \mathcal{C} \to \mathcal{C}'$ be chain maps. Then a chain homotopy of ϕ to ψ is a family of hom's

$$D_p : C_p \to C'_{p+1} \quad \text{s.t.}$$

$$\partial D + D\partial = \psi - \phi .$$

We write $D(\phi) = \psi$.

Def A chain map $\phi : \mathcal{C} \to \mathcal{C}'$ is a chain equivalence if $\exists\ \phi' : \mathcal{C}' \to \mathcal{C}$ s.t. $\phi' \circ \phi$ and $\phi \circ \phi'$ are chain homotopic to the identity maps on $\mathcal{C}$ and $\mathcal{C}'$, resp. ϕ' is a chain-homotopy inverse to ϕ.

Five Facts These are listed on pg 73. The proofs are left to the exercises. I'll just state them in brief

(1) Ch. homotopy is an eq. rel.
(2) Composition of ch. maps is well defined.
(3) Ch. hom. ch maps induce the same hom's on the homology gps.
(4) If ϕ, ϕ' are ch. eq. inverses their induced hom's on homology gps are isomorphisms that are inverses to each other.
(5) If $\phi : \mathcal{C} \to \mathcal{C}'$ and $\phi' : \mathcal{C}' \to \mathcal{C}''$ are ch. eq's, then $\psi \circ \phi$ is a ch. eq.

Lemma 13: Let $\mathcal{C}$ and $\mathcal{C}'$ be non-neg. ch. complexes.
Let $\varphi, \psi : \mathcal{C} \to \mathcal{C}'$ be ch. maps.
Let D be a ch. homotopy of ψ to φ.

① If φ is aug.-pres. $(\varepsilon' \circ \varphi_0 = \varepsilon)$, then so is ψ, $(\varepsilon' \circ \psi_0 = \varepsilon)$.

② ~~If φ preserves augmentation,~~
If we extend φ and ψ to be the identity $: \mathbb{Z} \to \mathbb{Z}$ in dimension -1, and define $D_{-1} : \mathbb{Z} \to C_0'$ to be the zero map, the D becomes a ch. homotopy between the extended map.

Picture

$$\begin{tikzcd}
\cdots & 0 \arrow[l] & 0 \arrow[l] & C_0' \arrow[l, "\partial_0'"'] & C_1' \arrow[l, "\partial_1'"'] & C_2' \arrow[l, "\partial_2'"'] & C_3' \arrow[l, "\partial_3'"'] & \cdots \arrow[l] \\
\cdots & 0 \arrow[l] & 0 \arrow[l] & C_0 \arrow[l, "\partial_0"] \arrow[u, shift left, "\psi_0"'] \arrow[u, shift right, "\varphi_0"] \arrow[ur, "D"] & C_1 \arrow[l, "\partial_1"] \arrow[u, shift left, "\psi"'] \arrow[u, shift right, "\varphi"] \arrow[ur, "D"] & C_2 \arrow[l, "\partial_2"] \arrow[u, shift left] \arrow[u, shift right] \arrow[ur] & C_3 \arrow[l, "\partial"] \arrow[u, shift left] \arrow[u, shift right] & \arrow[l]
\end{tikzcd}$$

where $\partial D + D\partial = \varphi - \psi$.

$$\begin{tikzcd}
\cdots & 0 \arrow[l] & \mathbb{Z} \arrow[l] & C_0' \arrow[l, "\varepsilon'"'] & C_1' \arrow[l, "\partial'"'] & C_2' \arrow[l, "\partial'"'] & C_3' \arrow[l, "\partial'"'] & \cdots \arrow[l] \\
\cdots & 0 \arrow[l] & \mathbb{Z} \arrow[l] \arrow[u, "id"] \arrow[ur, "zero"] & C_0 \arrow[l, "\varepsilon"] \arrow[u, shift left] \arrow[u, shift right] \arrow[ur] & C_1 \arrow[l, "\partial"] \arrow[u, shift left] \arrow[u, shift right] & C_2 \arrow[l, "\partial"] \arrow[u, shift left] \arrow[u, shift right] & C_3 \arrow[l, "\partial"] \arrow[u, shift left] \arrow[u, shift right] & \cdots \arrow[l]
\end{tikzcd}$$

Proof ① We need to show $\varepsilon'\psi = \varepsilon$. Let $c \in C_0$. By def

$$\partial_1' D c + D_{-1}\partial_0 c = \varphi_0(c) - \psi_0(c).$$

Since $\partial_0 c = 0$, $\quad \partial_1' D_0 c = \varphi_0(c) - \psi_0(c)$.

Since $\varepsilon'\partial_1' = 0$, $\quad \varepsilon'(\partial_1' D_0 c) = 0$. Thus

$$\begin{aligned} 0 &= \varepsilon'(\varphi_1(c) - \psi_0(c)) \\ &= \varepsilon'\varphi_0(c) - \varepsilon'\psi_0(c) \\ &= \varepsilon(c) - \varepsilon'\psi_0(c). \end{aligned}$$

Thus,

$\varepsilon'\psi = \varepsilon$ as claimed.

② All we need to show is

Ⓐ $\partial_1' D_0 + D_{-1}\varepsilon = \varphi_0 - \psi_0$

Ⓑ $\varepsilon' D_{-1} + D_{-2}\partial_{-1} = \varphi_{-1} - \psi_{-1}$.

Pf of Ⓐ Let $c \in C_0$. $D_{-1}\varepsilon(c) = 0$ since $D_{-1}: \mathbb{Z} \to 0 \in C_0$.
Then $\partial_1' D_0(c) = \varphi_0(c) - \psi_0(c)$ as before.

Pf of Ⓑ $D_{-2}: 0 \to \mathbb{Z}$ is trivial. Let $n \in \mathbb{Z}$.

$(\varepsilon' D_{-1} + D_{-2}\partial_{-1})(n) = \varepsilon'(0) = 0$.
$\varphi_{-1}(n) - \psi_{-1}(n) = id(n) - id(n) = n - n = 0$. $\square$

Lem 13.2 Let $\mathcal{C}$ and $\mathcal{C}'$ be non-neg. chain complexes.
Let $\mathcal{C}_\varepsilon$ and $\mathcal{C}'_\varepsilon$ be augmentations.
Let $\varphi: \mathcal{C} \to \mathcal{C}'$ be a chain equivalence with chain-homotopy inverse φ'. ~~Then~~
Then:

1. If φ pres. aug., so does φ'.
2. φ and φ' can be extended to a chain eq of $\mathcal{C}_\varepsilon$ and $\mathcal{C}'_\varepsilon$.
3. ~~(3)~~

Therefore, φ and φ' induce isomorphisms, that are inverses of each other, on the reduced homology groups.

Pf: See textbook.

Back to complexes.

Def Let K and L be complexes. An acyclic carrier from K to L is a function Φ from simplices of K to subcomplexes of L s.t.

① $\Phi(\sigma)$ is nonempty and acyclic $\forall$ simplices σ.
② If s is a face of σ, then $\Phi(s) \subset \Phi(\sigma)$.

Ex The opperation $\sigma \mapsto L(\sigma)$ in the proof of Thm 12.5 is the motivating example.

Def A hom. $f: C_p(K) \to C_q(L)$ is carried by Φ if for each p-simplex of K, $f(\sigma)$ is carried by $\Phi(\sigma)$.

Thm 13.3 Let Φ be an acyclic carrier from K to L.
ⓐ If ϕ, ψ are aug-pres. chain maps $\mathcal{C}(K) \to \mathcal{C}(L)$ that are carrier by Φ, then $\exists$ a chain homotopy D of ϕ to ψ that is carried by Φ.

ⓑ $\exists$ an aug-pres. ch. map from $\mathcal{C}(K)$ to $\mathcal{C}(L)$ that is carried by Φ.

Pf (a) Same as 12.5

(b) Consider the diagram, where $C_p = C_p(K)$ and $C_p' = C_p(L)$.

$$
\begin{tikzcd}
 & & 0 \arrow[dl] & & & & & \\
 & 0 \arrow[l] & \mathbb{Z} \arrow[l] & C_0 \arrow[l, "\varepsilon"] \arrow[ul] \arrow[d, "\varphi_0"] & C_1 \arrow[l] \arrow[d, "\varphi_1"] & C_2 \arrow[l] \arrow[d, "\varphi_2"] & C_3 \arrow[l] & \arrow[l] \\
 & 0 \arrow[l] & 0 \arrow[l] & C_0' \arrow[l] \arrow[dl, "\varepsilon'"] & C_1' \arrow[l] & C_2' \arrow[l] & C_3' \arrow[l] & \arrow[l] \\
 & & \mathbb{Z} \arrow[ul] & & & & &
\end{tikzcd}
$$

$p=0$ For each vertex v of K ($\in C_0$) define $\varphi_0(v) = v'$ where v' is any vertex of $\Phi(v)$. Extend to all of C_0. It is trivial to check that

$$
\begin{tikzcd}
0 \arrow[d, equal] & C_0 \arrow[l] \arrow[d, "\varphi_0"] \\
0 & C_0' \arrow[l]
\end{tikzcd}
$$

commutes.

What about

$$
\begin{tikzcd}
\mathbb{Z} \arrow[d, equal] & C_0 \arrow[l, "\varepsilon"'] \arrow[d, "\varphi_0"] \\
\mathbb{Z} & C_0' \arrow[l, "\varepsilon'"']
\end{tikzcd}
$$

?

For $v \in C_0$, $\varepsilon(v) = 1$ so $\varepsilon'(\varphi_0(v)) = \varepsilon'(v') = 1$.

Thus $\sum n_i v_i \in C_0$, $\varepsilon'(\varphi_0(\sum n_i v_i)) = \varepsilon'(\sum n_i v_i') = \sum n_i$
$= \varepsilon(\sum n_i v_i)$.

$p=1$ Let $\sigma = [v, w] \in C_1$. We want to define $\varphi_1(\sigma) \in C_1'$.

Consider $c = \varphi_0(\partial\sigma) \in C_0'$. We are going to show $\exists\, d \in C_1'$ s.t. $\partial d = c$. Then we define $\varphi_1(\sigma) = d$. It is then automatic that $\partial\varphi = \varphi\partial$.

Claim c is carried by $\Phi(\sigma)$.

Pf: $\phi_0(\partial_1\sigma) = \phi_0(w) - \phi_0(v)$. This is carried by $\Phi(v) \cup \Phi(w)$. But these are in $\Phi(\sigma)$ by ② in definition of acyclic carrier.

Claim c represents an element of $\tilde{H}(\Phi(\sigma))$.

Pf: $\varepsilon'(c) = \varepsilon'(\phi_0(\partial\sigma)) = \varepsilon(\partial\sigma) = 1 - 1 = 0$.
Then $c \in \tilde{Z}_0(\Phi(\sigma))$.

Since $\tilde{H}_0(\Phi(\sigma)) = 0$ by (1), $\exists$ 1-chain $d \in C_1(\Phi(\sigma)) \subset C_1(L)$ s.t. $\partial d = c$.

Let $\phi_1(\sigma) = d$, Extend to C_1. $\partial\phi = \phi\partial$ is clear.

$p > 1$ The induction step is similar. See text. $\blacksquare$

Now we go to a more abstract viewpoint.

Def Let $\mathcal{C} = \{(C_p, \partial_p)\}$ be a ch. complex. A subchain complex $\mathcal{B} = \{(B_p, \partial_p^B)\}$ is a ch. complex with B_p a subgp of C_p and $\partial_p^B = \partial_p|B$.

Def Let $\mathcal{C}$ be a free, non-neg. ch. complex with aug. $\mathcal{C}_\varepsilon$. For each C_p specify a basis $\{\sigma_p^\alpha\}_{\alpha \in J_p}$.

Let $\mathcal{C}'$ be another non-neg ch. complex with aug. $\mathcal{C}'_{\varepsilon'}$.

A acyclic carries from $\mathcal{C}$ to $\mathcal{C}'$, relative to the given basis is a function Φ that takes each σ_p^α to a subchain complex $\Phi(\sigma_p^\alpha)$ of $\mathcal{C}'$ s.t.

① $\Phi(\sigma_p^\alpha)$ is acyclic and can be augmented by ε'.

② If σ_{p-1}^β is in the expression for $\partial_p \sigma_p^\alpha$, is terms of the given basis of C_{p-1}, then $\Phi(\sigma_{p-1}^\beta)$ is a subchain complex of $\Phi(\sigma_p^\alpha)$.

A hom. $f : C_p \to C'_q$ is carried by Φ if

$$f(\sigma_p^\alpha) \in C'_q(\Phi(\sigma_p^\alpha)) \subset C'_q \quad \forall \alpha.$$

Thm (13.4) The analog of Thm 13.3 holds. See textbook for proof.

Ordered Simplicial Homology

Example

$K = \{ v_0, v_1, v_2, v_0v_1, v_1v_2 \}$

There is no orientation assigned to the members of K. In oriented simp. homology we assign an orientation to each edge & let the chain groups be

$$C_0 = \langle v_0, v_1, v_2 \rangle \quad C_1 = \langle [v_0v_1], [v_1v_2] \rangle.$$

where $[v_0v_1]$ and $[v_1v_2]$ are equivalence classes of even permutations (trivial in one-dim.) and the odd permutations are regarded as their inverses.

Now we define a different homology theory on K called ordered simplicial homology. Let $C_0' = \langle v_0, v_1, v_2 \rangle$ as before, but

$$C_1' = \langle v_0v_0, v_0v_1, v_1v_0, v_1v_1, v_1v_2, v_2v_1, v_2v_2 \rangle$$

= the free abelian gp generated by all pairs of vertices which are on a common 1-simplex of K.

In general, C_p' = free ab. gp generated by all $p+1$ tuples of vertices which are on a common p-simplex of K.

Define: $\partial_1'(v_iv_j) = v_j - v_i$

$$\partial_p'(v_0 \ldots v_p) = \sum_{i=0} (-1)^i (v_0 \ldots \hat{v}_i \ldots v_p)$$

As before $\partial \circ \partial = 0$

Let's find $H_0(k)$.

$$C_0' = \langle v_0 v_1 v_2 \rangle \cong \mathbb{Z}^3$$
$$C_1' = \langle v_0 v_0, \ldots, v_2 v_2 \rangle \cong \mathbb{Z}^7.$$

$H_0(k) = \dfrac{\ker \partial_0'}{\operatorname{im} \partial_1'}$. $\ker \partial_0' = C_0'$.

$\partial_1' v_0 v_0 = v_0 - v_0 = 0$
$\partial_1' v_0 v_1 = v_1 - v_0$
$\partial_1' v_1 v_0 = v_0 - v_1$ $\qquad \operatorname{im} \partial_1' = \langle v_0 - v_1, v_1 - v_2 \rangle$
$\partial_1' v_1 v_1 = 0$
$\partial_1' v_1 v_2 = v_2 - v_1$
$\partial_1' v_2 v_1 = v_1 - v_2$
$\partial_1' v_2 v_2 = 0$

$$H_0(k) = \frac{\langle v_0, v_1, v_2 \rangle}{\langle v_0 - v_1, v_1 - v_2 \rangle} = \frac{\langle v_0 - v_1, v_1 - v_2, v_2 \rangle}{\langle v_0 - v_1, v_1 - v_2 \rangle} \cong \mathbb{Z}.$$

Lem 13.5 Let $w * K$ be a cone over the complex K. Then $w * K$ is acyclic in the ordered homology.

Pf Define $D : C_p'(w*K) \to C_{p+1}'(w*K)$ by

$$D\big((v_0, \ldots, v_p)\big) = (w, v_0, \ldots, v_p).$$

Any of the v_i could be w

Let $c_0 \in C_0(w*K)$, $c_0 = \sum n_i v_i$.

$$\partial_1' D c_0 = \sum n_i \partial_1' D v_i = \sum n_i \partial_1' (w v_i)$$

$$= \sum n_i (v_i - w) = \sum n_i v_i - \left(\sum n_i\right) w$$

$$= c_0 - \varepsilon'(c_0) w.$$

$p=0$ If $c_0 \in \ker \varepsilon'$, then $\varepsilon'(c_0) = 0$ and so $c_0 \in \operatorname{im} \partial_1'$. Thus, $\tilde{H}_0(w*K) = 0$.

$p>0$ Let $c_p \in C_p(w*K)$, $c_p = \sum n_i \sigma_i$.
Just look at a single p-simplex σ.

$$\partial_{p+1}' D\sigma = \partial_{p+1}'(w v_0 \ldots v_p) = (v_0 \ldots v_p) - (w v_1 \ldots v_p)$$
$$+ (w v_0 v_2 \ldots v_p)$$
$$\vdots$$
$$\pm (w v_0 \ldots v_{p-1})$$

$$= \sigma - D\partial_p' \sigma$$

Thus,

$$\partial_{p+1}' D c_p = c_p - D \partial_p' c_p.$$

Now suppose c_p is a cycle, $\partial c_p = 0$. Then

$$\partial'_{p+1} D c_p = c_p - D \partial'_p c_p = c_p.$$

Thus $c_p \in \operatorname{im} \partial'_{p+1}$. Hence $\tilde{H}'_p(w * K) = 0$.

$\square$

Thm 13.6 Let K be a complex. Then $H'_p(K) \cong H_p(K)$.

Outline Pf Choose a partial ordering of the vertices of K that induces a linear ordering on the vertices of each simplex of K.

Define

$$\phi : C_p(K) \to C'_p(K)$$

by

$$\phi([v_0 \cdots v_p]) = (v_0 \cdots v_p)$$

if $v_0 < v_1 < \cdots < v_p$.

Define $\psi : C'_p(K) \to C_p(K)$ by

$$\psi(w_0 \cdots w_p) = \begin{cases} [w_0 \cdots w_p] & \text{if } w_i \text{ are distinct} \\ 0 & \text{otherwise.} \end{cases}$$

Then show ϕ and ψ are aug-pres. ch. maps.

That are ch. homotopy inverses.

Apply Thm 12.4.